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Multipole Inverse Support Theory

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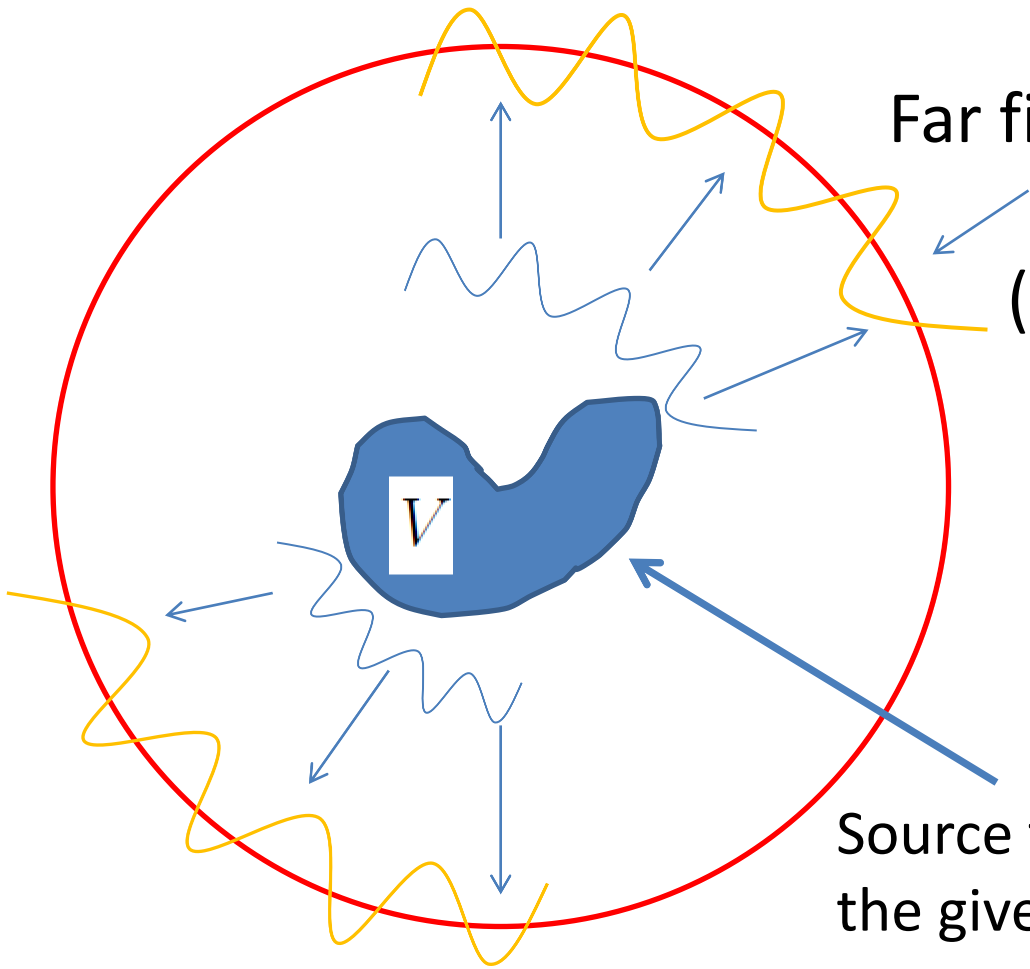
Pictorial Problem Description



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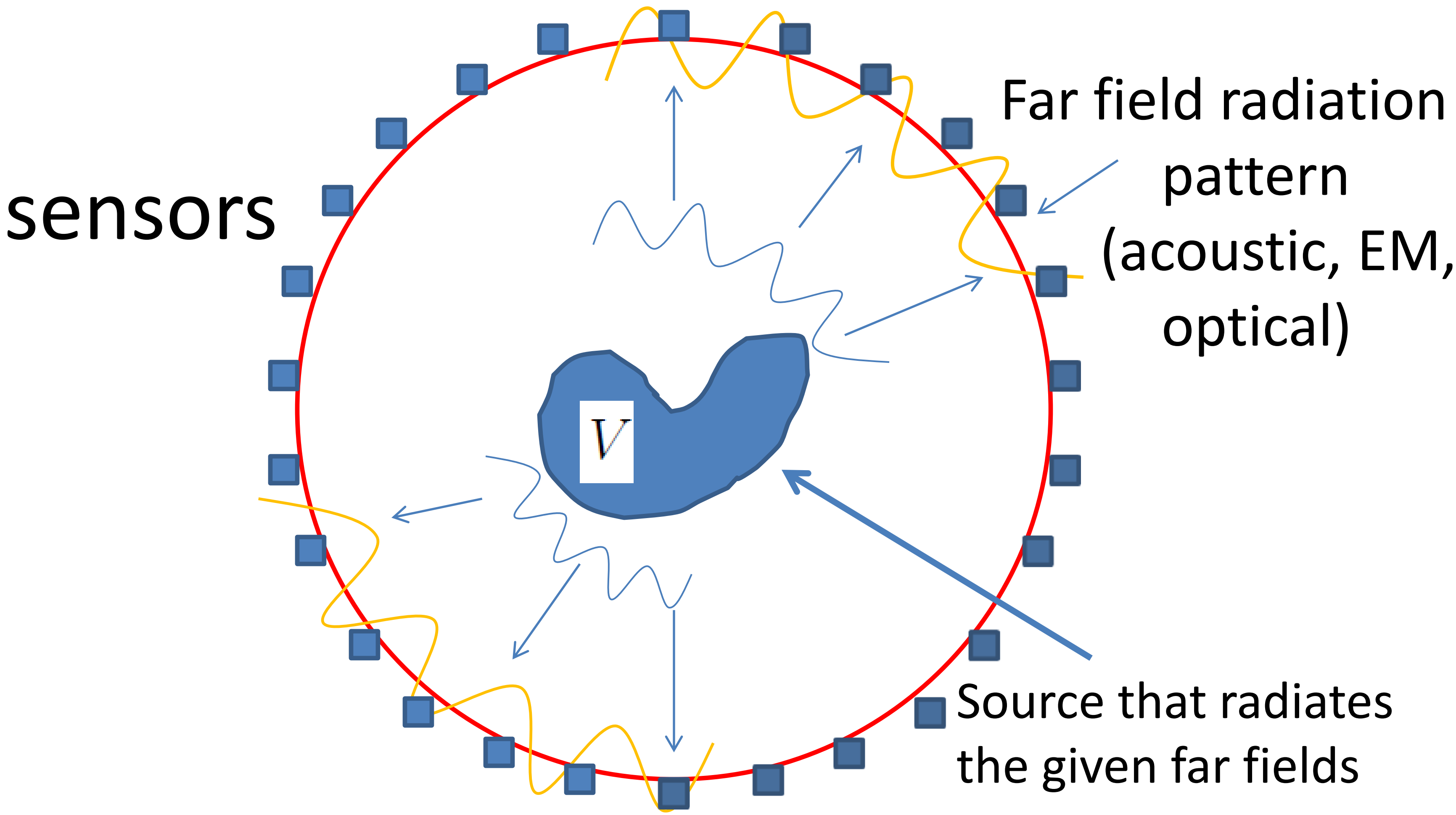
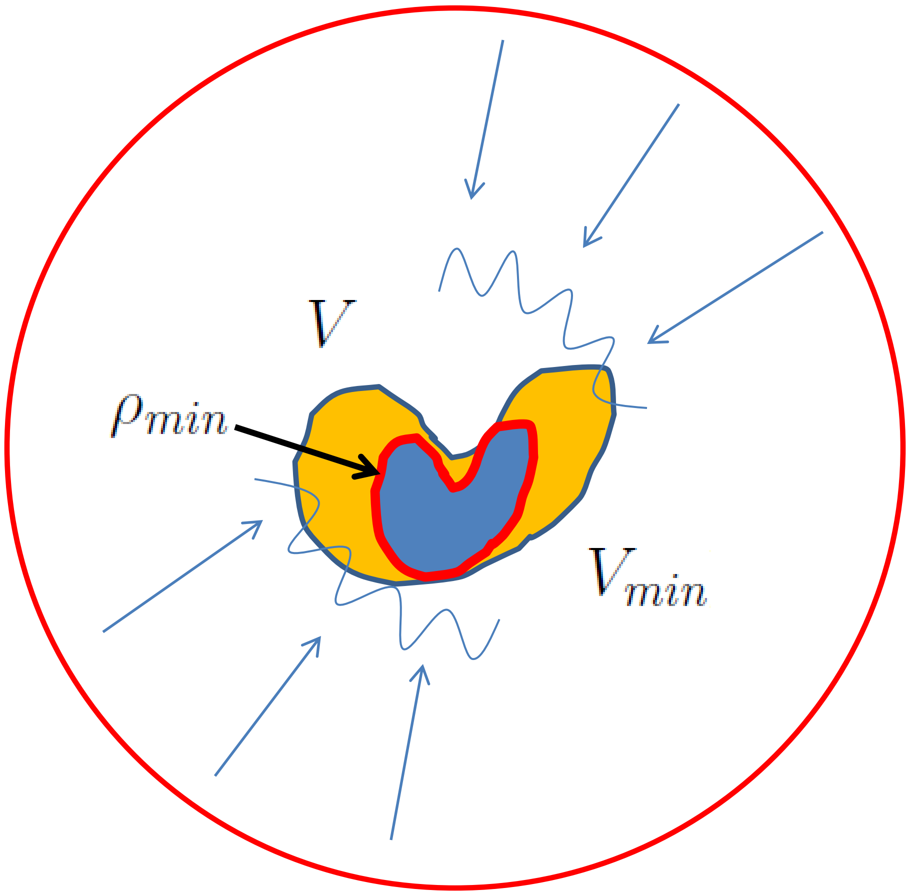


What is the Minimum Support?



- Note: true (original) support contains the minimum region

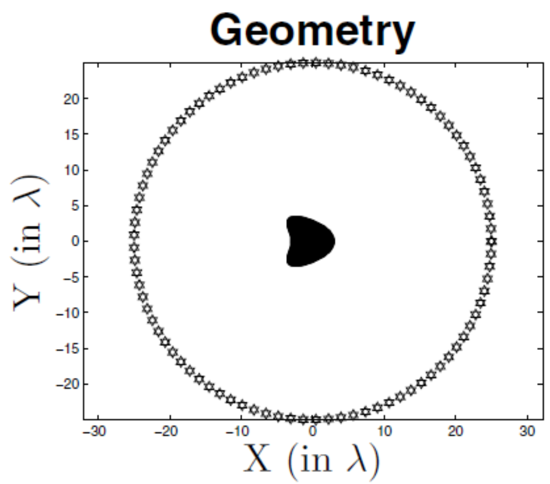
- There exists a source in the minimum region that produces the field

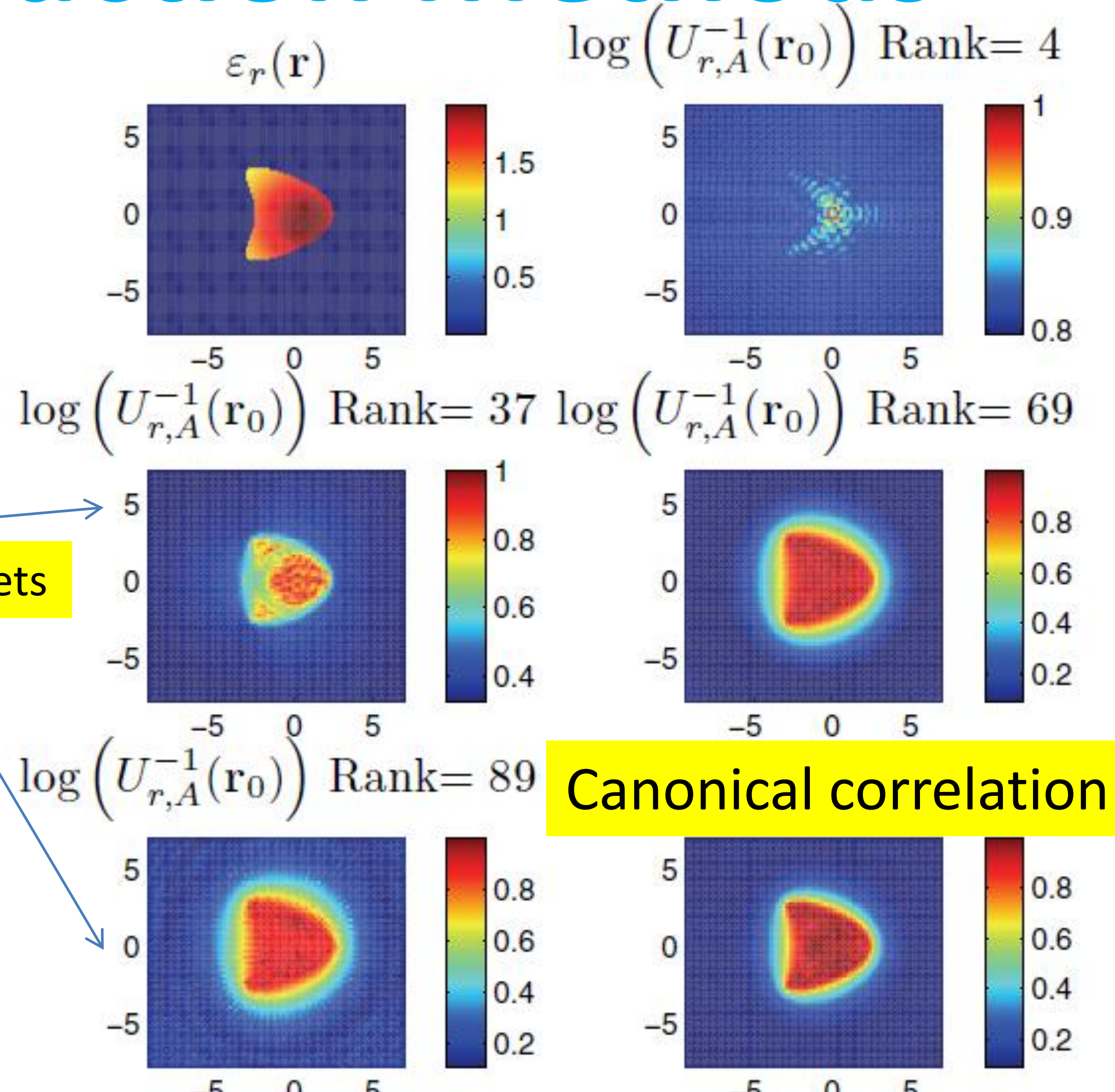
(sparsest solution)

- Largest 2 norm of all sources that produce the field and lack nonrad. part

Solution essentially defines the 'center' of a far field (is a 'common denominator').

Many Imaging and Shape Reconstruction Methods





MUSIC for extended targets

Gruber and Marengo, Reinterpretation and enhancement of signal subspace based methods for extended scatterers, SIAM J. Imag. Sciences, 2010.

Some Questions

- Classical representation counterpart of the qualitative Imaging methods

- Theoretical explanation of the observed robustness of the qualitative imaging methods, and the fundamental limits

- Understanding of the limits of the limited data inverse source and scattering reconstructions (compressive sensing); single-transmit experiment

Two Lines of Research

- New methodology to estimate **minimum source regions** of far fields, and bounds for those regions.

*Multipole-based algorithms with linkages to backpropagation

- Minimum convex source regions (subset of the convex hull)
- Minimum source region which can be nonconvex

- Minimum source regions of far fields (and their bounds) are **probabilistically robust bounds** for the true supports of sources and scatterers.

*Probabilistic understanding of observed inversion robustness

- Signal subspace considerations give **confidence intervals** for the estimated minimum source or scatterer support.

Applications of Inverse Support Theory

- Minimum source region of a far field is **unique** despite the nonuniqueness of the inverse source problem.
- Represents **localization information** contained in the far field.
- **Analytical formulas** for closed-form field data offer rigor to the inverse support theory.
- Constitutes part of a purely analytical approach to solve certain canonical inverse source and scattering problems, relevant to imaging and antenna theory. (noniterative: JASA, 2006)
- **Computational version** of the approach is an inversion method for, e.g., shape reconstruction.
- **Universality**, due to reliance on a Picard test (e.g., inverse problems in optical coherence theory).

Literature and Contributions

- Existence: Muller (1969); we demonstrate the algorithms.
- Plane wave expansion closed-form approach: Yaghjian et al. (1997); we compute the multipole counterpart which expands class of problems tackled analytically.
- Paley-Wiener theorem for the convex scattering support: Kusiak and Sylvester (2003); we extend to nonconvex part.
- ‘Range test’ for practical application: Potthast et al. (2003). Picard test reinterpretation: Kusiak and Sylvester (2005); our convex and nonconvex support inversion approach is also a Picard test (defines class of realizable data in L2 constraint).
- Existence of UWSC sets: Sylvester (2006); we demonstrate concrete examples of the UWSC sets (derived analytically).
- Extension to backscattering (radar): Haddar et al. (2005).

Two Approaches

- Radiation (maps of sources to fields)
- Diffraction (maps of fields to fields)
- They are found to be equivalent
- Results in both 3D and 2D spaces.

Scalar Focus With Vector Extension

- Scalar waves (Helmholtz)
- The theory uses multipole expansions, and asymptotic properties of spherical Bessel functions.
- Diffraction form of the theory **holds also for the full vector (EM) case** (or use vector multipole theory within the radiation form).

The Helmholtz equation

Let us consider fields obeying the Helmholtz equation

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = -\rho(\mathbf{r}) \tag{1}$$

(+ radiation condition)

where $k = \omega/c$ is the wavenumber and ρ the source function.

The source-based representations take alternative forms, starting with Green's function representation

$$\psi(\mathbf{r}) = \int_V d\mathbf{r}' G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}'). \tag{2}$$

Active source (emitter) or induced source (scatterer)

Green's Function (3D)

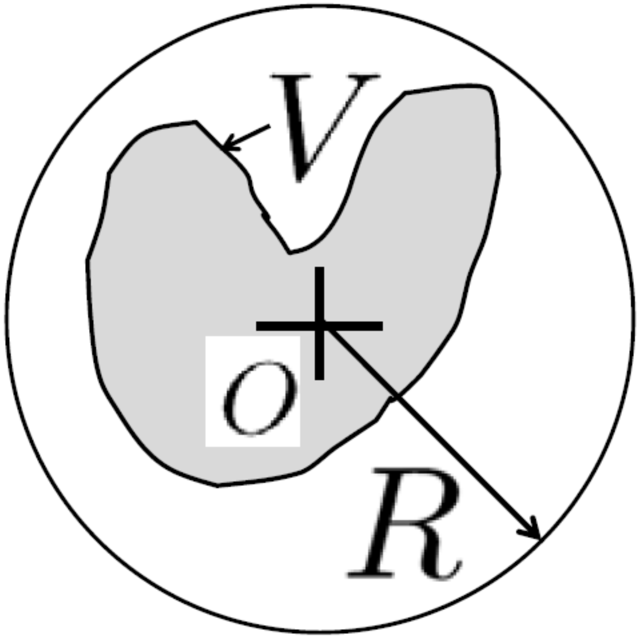
$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

which via addition theorem for the spherical Hankel function becomes

$$\boxed{G(\mathbf{r}, \mathbf{r}') = ik \sum_{l=0}^{\infty} \sum_{m=-l}^{l} j_l(kr_<) h_l^{(1)}(kr_>) Y_{l,m}(\hat{\mathbf{r}}) Y_{l,m}^*(\hat{\mathbf{r}}')}$$

where $r_< = \min(r, r')$ (where $r' \equiv |\mathbf{r}'|$) and $r_> = \max(r, r')$.

Multipole Expansion



For this origin, source is contained in ball of radius R

$$\psi(\mathbf{r}) = ik \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m} h_l^{(1)}(kr) Y_{l,m}(\hat{\mathbf{r}}), \quad r > R$$

Far Fields and Multipole Moments

In far zone, $\boxed{\psi(r\hat{\mathbf{r}}) \sim f(\hat{\mathbf{r}})\frac{\exp(ikr)}{r}}$

The generated far field radiation pattern

$$f(\hat{\mathbf{r}}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l a_{l,m} Y_{l,m}(\hat{\mathbf{r}}) \tag{7}$$

where $Y_{l,m}$ is the spherical harmonic of degree l and order m and the multipole moments

Linear projections

$$\boxed{a_{l,m} = \int_V d\mathbf{r}\, j_l(kr) Y_{l,m}^*(\hat{\mathbf{r}}) \rho(\mathbf{r})} \tag{8}$$

where $j_l(\cdot)$ is the spherical Bessel function of order l.

Plane Wave Representation

Thus in the far zone

$$\psi(r\hat{\mathbf{r}}) \sim \frac{\exp(ikr)}{4\pi r} f(\hat{\mathbf{r}}) \tag{3}$$

where

$$f(\hat{\mathbf{r}}) = \int_V d\mathbf{r}\rho(\mathbf{r}) \exp(-ik\mathbf{s}\cdot\mathbf{r}). \tag{4}$$

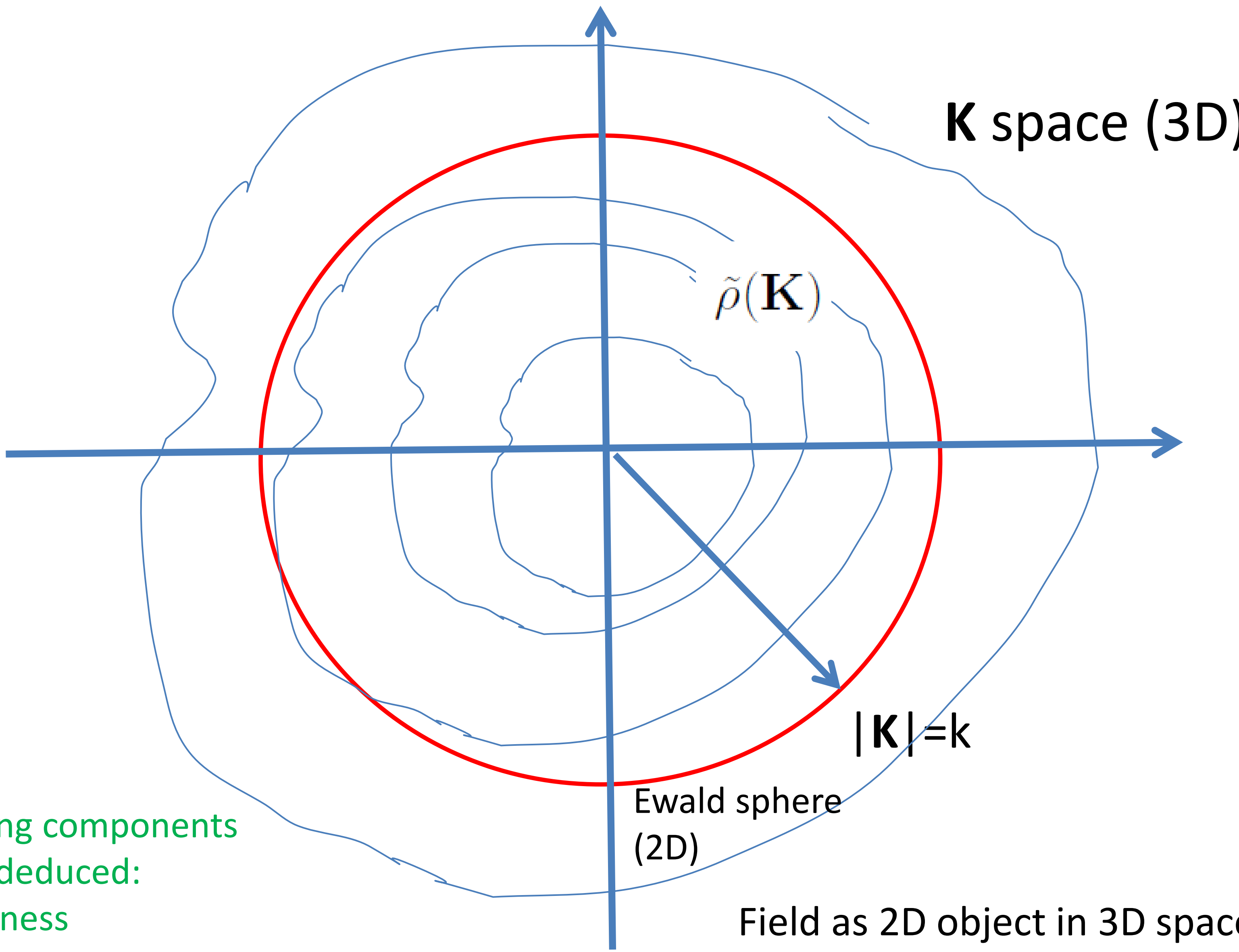
In Fourier representation

$$f(\hat{\mathbf{r}}) = \tilde{\rho}(\mathbf{K} = k\mathbf{s}) \quad \mathbf{s} \in S^2 \tag{5}$$

where

$$\tilde{\rho}(\mathbf{K}) = \int_V d\mathbf{r}\rho(\mathbf{r}) \exp(-i\mathbf{K}\cdot\mathbf{r}). \tag{6}$$

Information in Fourier Domain



Nonradiating components cannot be deduced: nonuniqueness

Field as 2D object in 3D space.

Built-in Incoherence

- Source-to-far-field mapping encodes information about the source in the form of:

projections of the source in a (Fourier exponential) basis that is ideally incoherent to the point-source or configuration space representation for the source.

- This mapping is an optimal form of compressive sensing.

Support is 3D, while far field info. Is 2D, yet it is in the form of an ideal projection

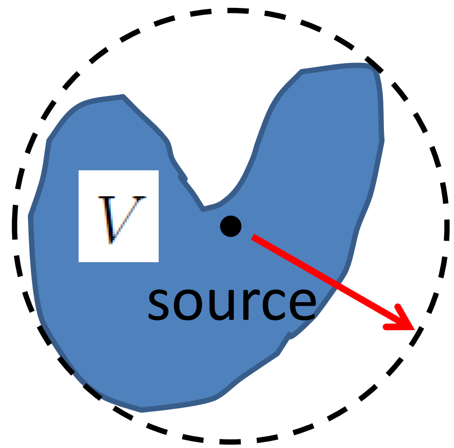
Sparsest Representation

This yields the sparse representation

$$\psi(\mathbf{r}) = \int_{V_{min}} d\mathbf{r}' \rho_{min}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \quad \mathbf{r} \notin V_{min} \tag{9}$$

consistent with the far field radiation pattern $f(\hat{\mathbf{r}})$.

Multipole Expansion



Smallest ball containing the source support

$$V \subseteq B_R \equiv \{\mathbf{r} \in \mathbb{R}^3 : r \leq R\}$$

$$\psi(r\hat{\mathbf{r}}) = ik \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m} h_l^{1} (kr) Y_{l,m}(\hat{\mathbf{r}}), \quad r > R \qquad (9)$$

Radiated Power

From the large argument approximation of the spherical Hankel function

$$f(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l a_{l,m} Y_{l,m}(\hat{\mathbf{r}})$$

so that due to orthonormality of the harmonics

$$a_{l,m} = i^l \int_{S^2} d\hat{\mathbf{r}} Y^*_{l,m}(\hat{\mathbf{r}}) f(\hat{\mathbf{r}}).$$

The radiated power is proportional to the far field L2 norm

$$||f||_2^2 = \int_{S^2} d\hat{\mathbf{r}} |f(\hat{\mathbf{r}})|^2 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |a_{l,m}|^2 < \infty.$$

Inverse Source Problem

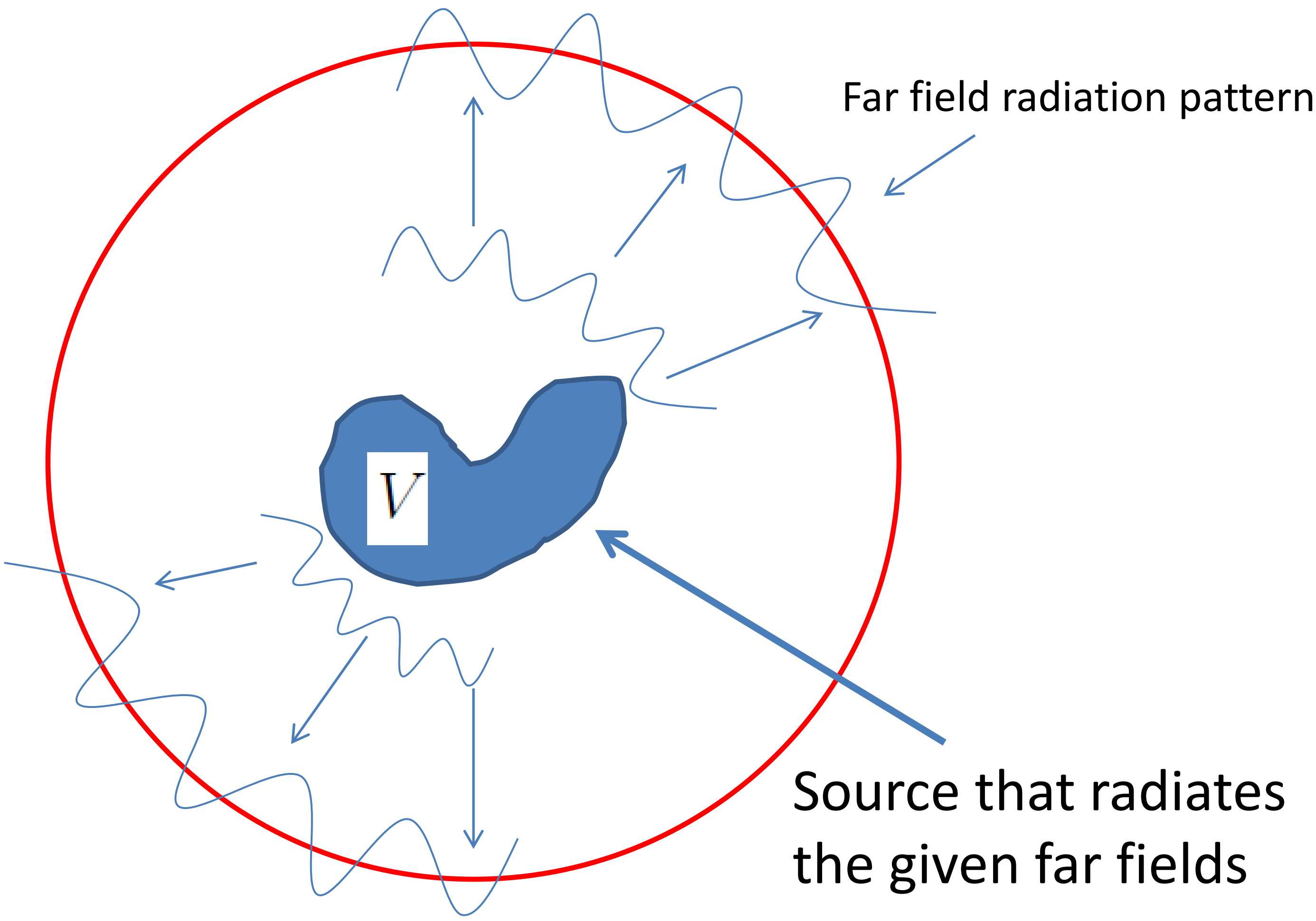
Inverse source problem with far field data: from $f(\hat{\mathbf{r}})$ determine $\rho(\mathbf{r})$.

- Nonuniqueness.

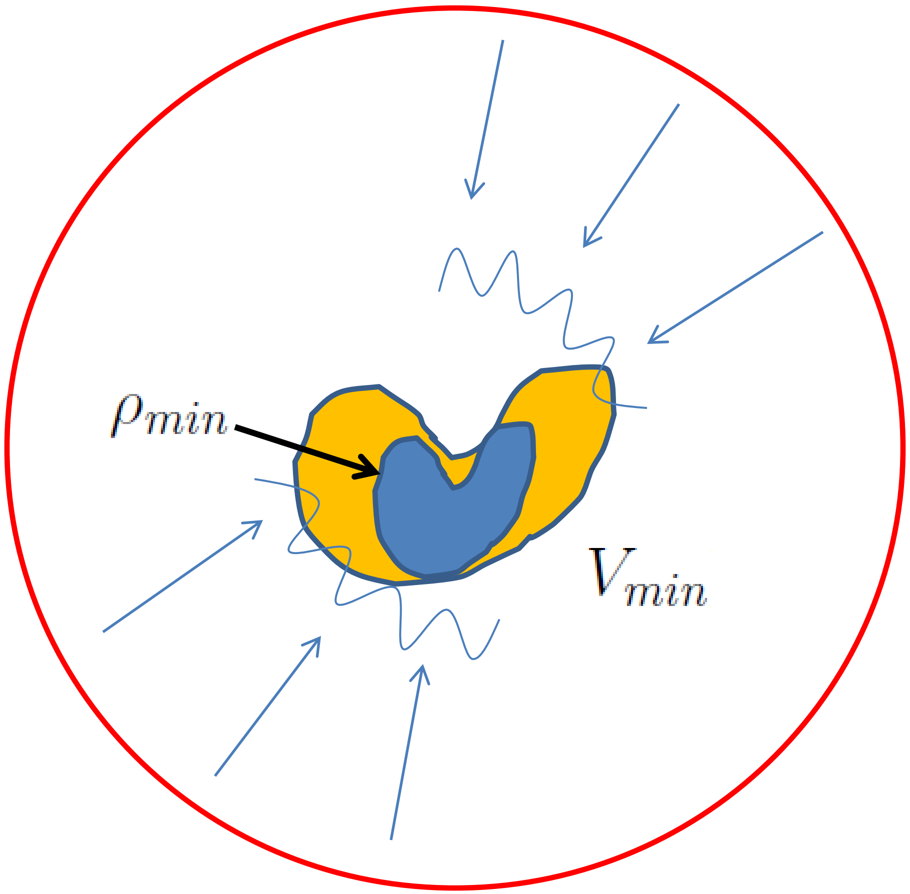
- can estimate unique minimum L^2 norm solution for given V.

- question: what is the smallest region V that can produce the field?

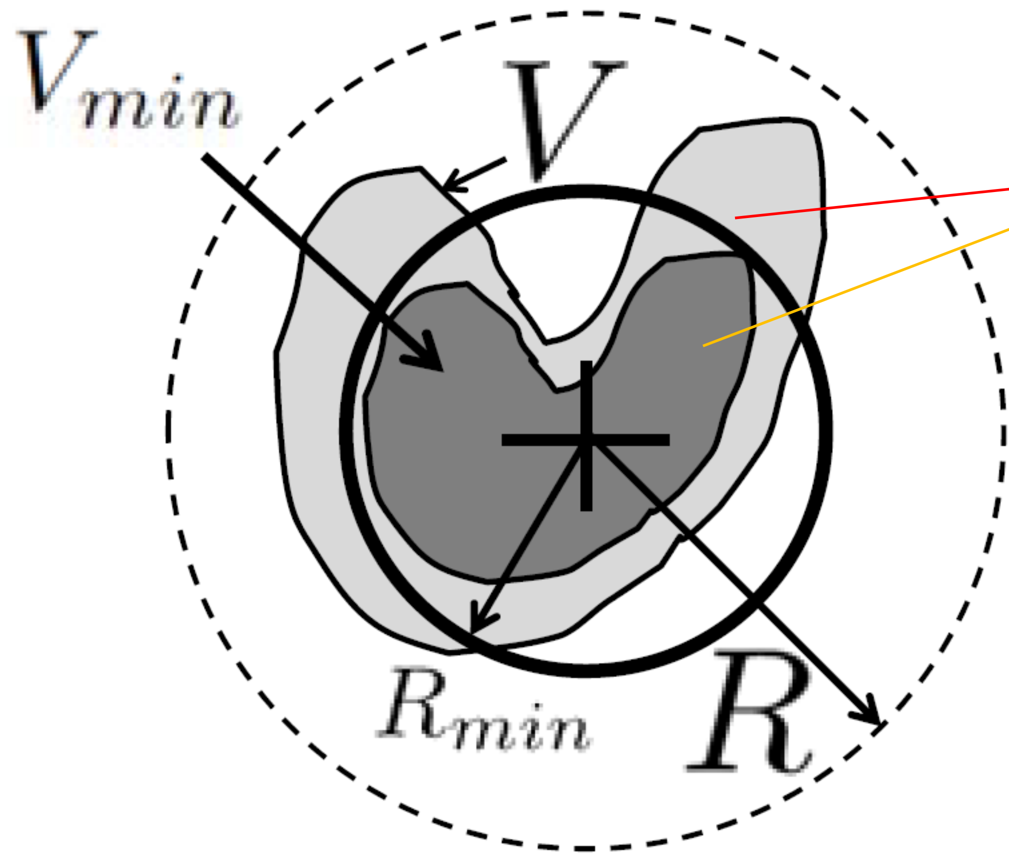
Original Source and its Field



Minimum Source Region



Relations



same field outside ball of radius R

$$\psi(\mathbf{r}) = ik \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m} h_l^{(1)}(kr) Y_{l,m}(\hat{\mathbf{r}}), \quad r > R_{min}$$

Minimum Energy Sources

$$a_{l,m} = \int_{r \le R} dr r^2 j_l(kr) \int_{S^2} d\hat{\mathbf{r}} Y^*_{l,m}(\hat{\mathbf{r}}) \rho(\mathbf{r})$$

These sources are of the form (minimum L2 norm solution)

$$\boxed{\hat{\rho}_a(\mathbf{r}) = M_{B_a}(\mathbf{r}) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{a_{l,m}}{\sigma_l^2(a)} j_l(kr) Y_{l,m}(\hat{\mathbf{r}})}$$

where $M_\tau(\mathbf{r}) = 1$ $(= 0)$ if $\mathbf{r} \in \tau$ (otherwise), and where

$$\sigma_l^2(a) = \int_{r \le a} dr r^2 j_l^2(kr) = \frac{a^3}{2}[j_l^2(ka) - j_{l-1}(ka) j_{l+1}(ka)].$$

$$B_a = \{\mathbf{r} \in \mathbb{R}^3 : r \le a\}$$

Question: How small can this radius (‘a’) be?

(Note: for nontrivial field, minimum source region is nonempty)

Asymptotic Behavior

Useful facts:

$$j_l(z) \sim \sqrt{\frac{\pi}{2z}} \frac{(z/2)^{l+1/2}}{\Gamma(l+3/2)}, \quad l \to \infty$$

from which it follows that

$$\sigma_l^2(a) \sim \frac{\pi a^3 (ka)^{2l}}{2^{2l+2}(2l+3)\Gamma^2(l+3/2)}, \quad l \to \infty$$

which means $\sigma_l^2(a) \to 0$ as $l \to \infty$.

The singular values $\sigma_l^2(a)$ decay exponentially fast for $l \gtrsim ka$.

Key difference between analytical versus computational

infinite finite

Minimum Radius

$$\boxed{||\hat{\rho}_a||^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} |a_{l,m}|^2/\sigma_l^2(a) < \infty.}$$

Picard condition which defines realizable fields.

(+ uniform, absolute conv.)

$$R_{min} = \sup\{a \in \mathbb{R}^+ : \lim_{l\to\infty} \frac{\sqrt{2l+3}\,\Gamma(l+3/2)}{(ka/2)^l}|a_{l,m}| \neq 0, \text{at least one } m\}$$

Equivalent!

Diffraction counterpart:

$$R_{min}(\alpha) = \sup\{b \in \mathbb{R}^+ : \lim_{l\to\infty} \frac{\Gamma(l+1/2)}{(kb/2)^l}|a_{l,m}| \neq 0, \text{at least one } m\}$$

(less stringent; square-integrability of surface field versus of volume source)

(Alternative conditions for physical reasonableness are all equivalent to this.)

Minimum Radius

It follows from the D'Alembert ratio test and the fact that

$$\lim_{l\to\infty}\frac{|a_{l+1,m}|^2\sigma_l^2(R_{min})}{|a_{l,m}|^2\sigma_{l+1}^2(R_{min})}=1$$

that

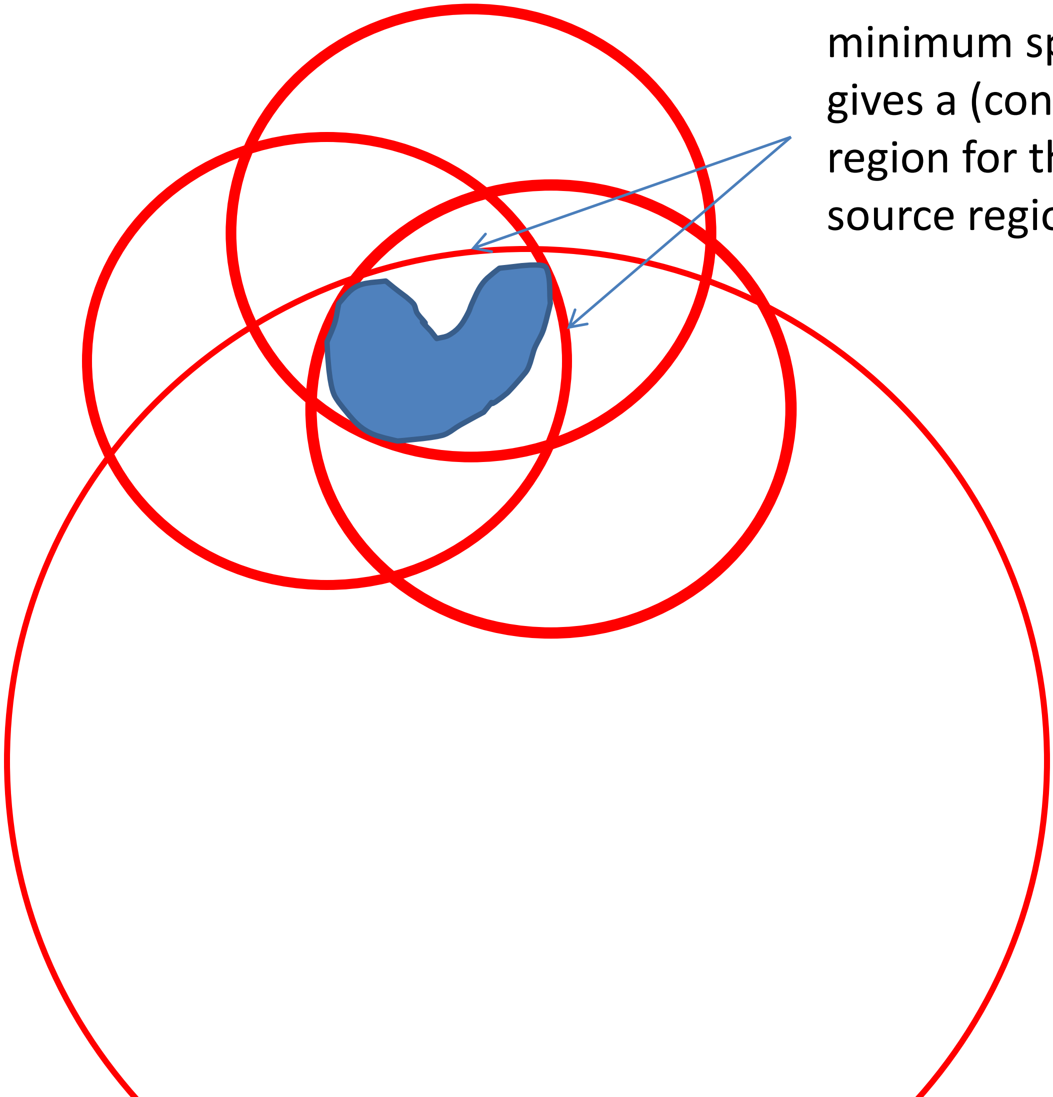
$$\lim_{l\to\infty}\frac{|a_{l+1,m}|^2\sigma_l^2(a)}{|a_{l,m}|^2\sigma_{l+1}^2(a)}>1 \quad \text{at least one } m \quad a<R_{min}$$

which gives

Holds for both open and closed ROC

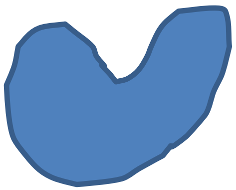
$$\lim_{l\to\infty}|a_{l,m}|^2/\sigma_l^2(a)\neq 0 \quad \text{at least one } n \quad a<R_{min}$$

Multiple Origins

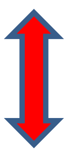


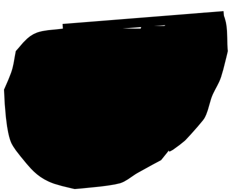
Intersection of the minimum spherical volumes gives a (convex) bounding region for the minimum source region.

Backpropagation

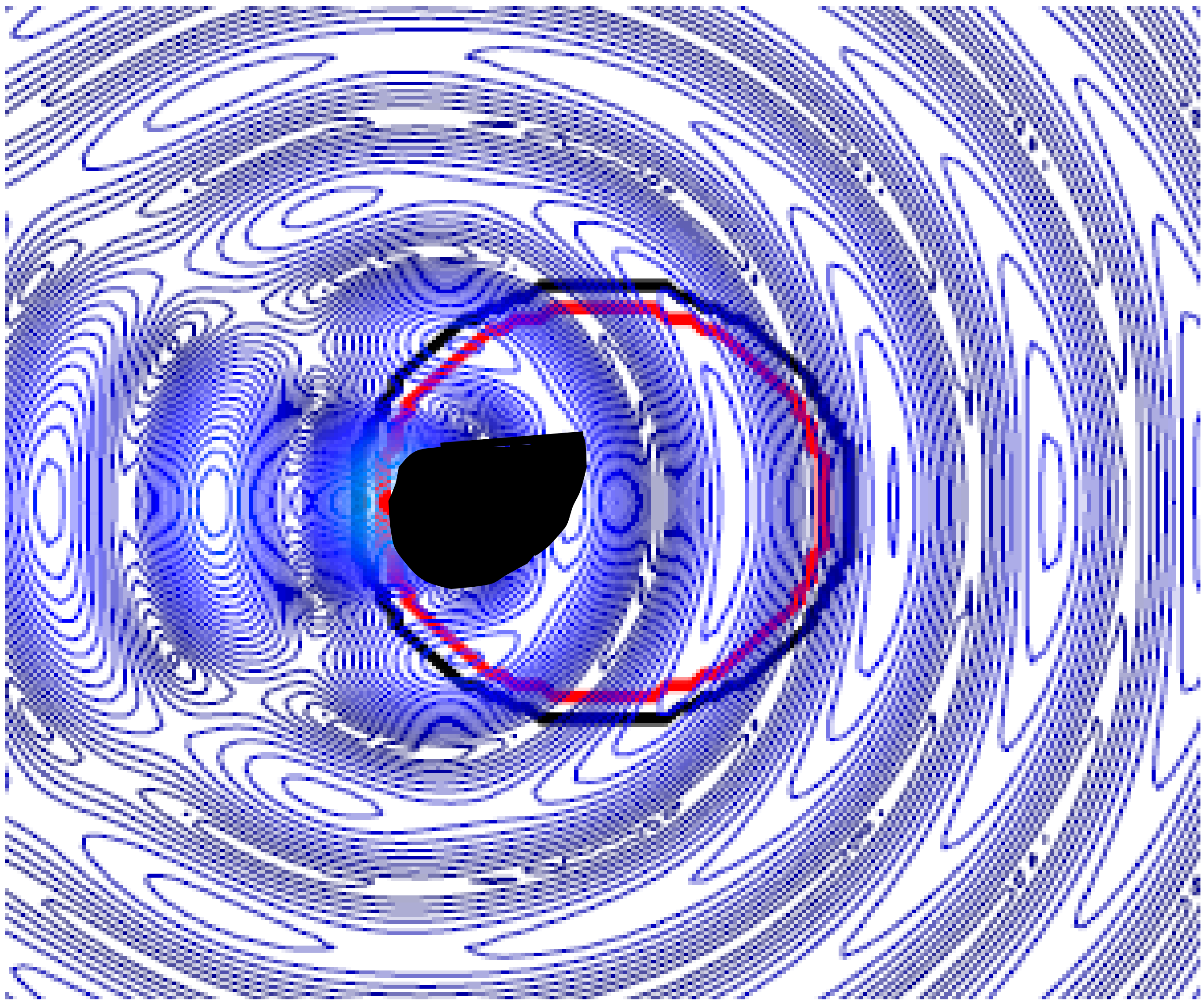


Min.
region



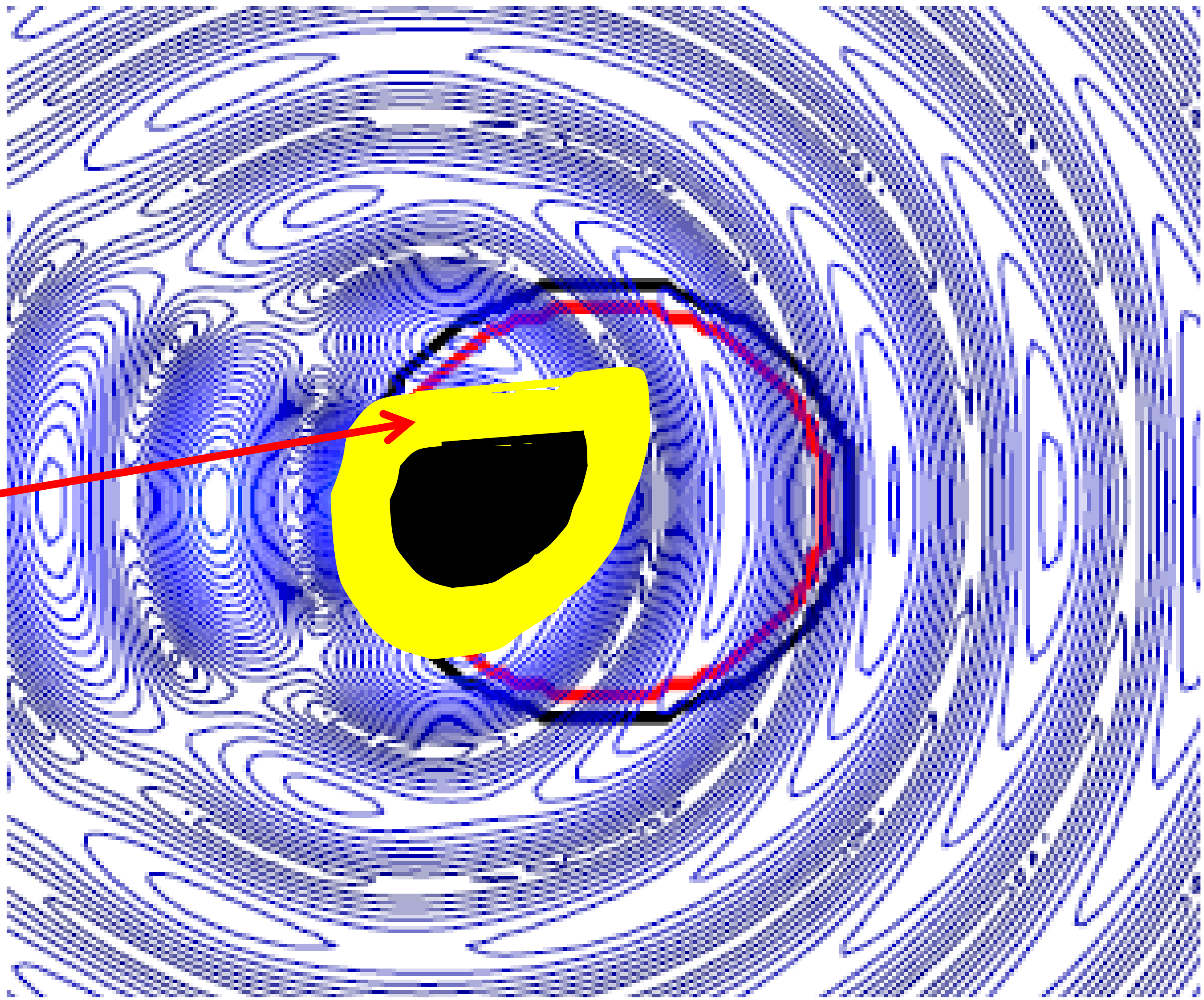


Convex
hull

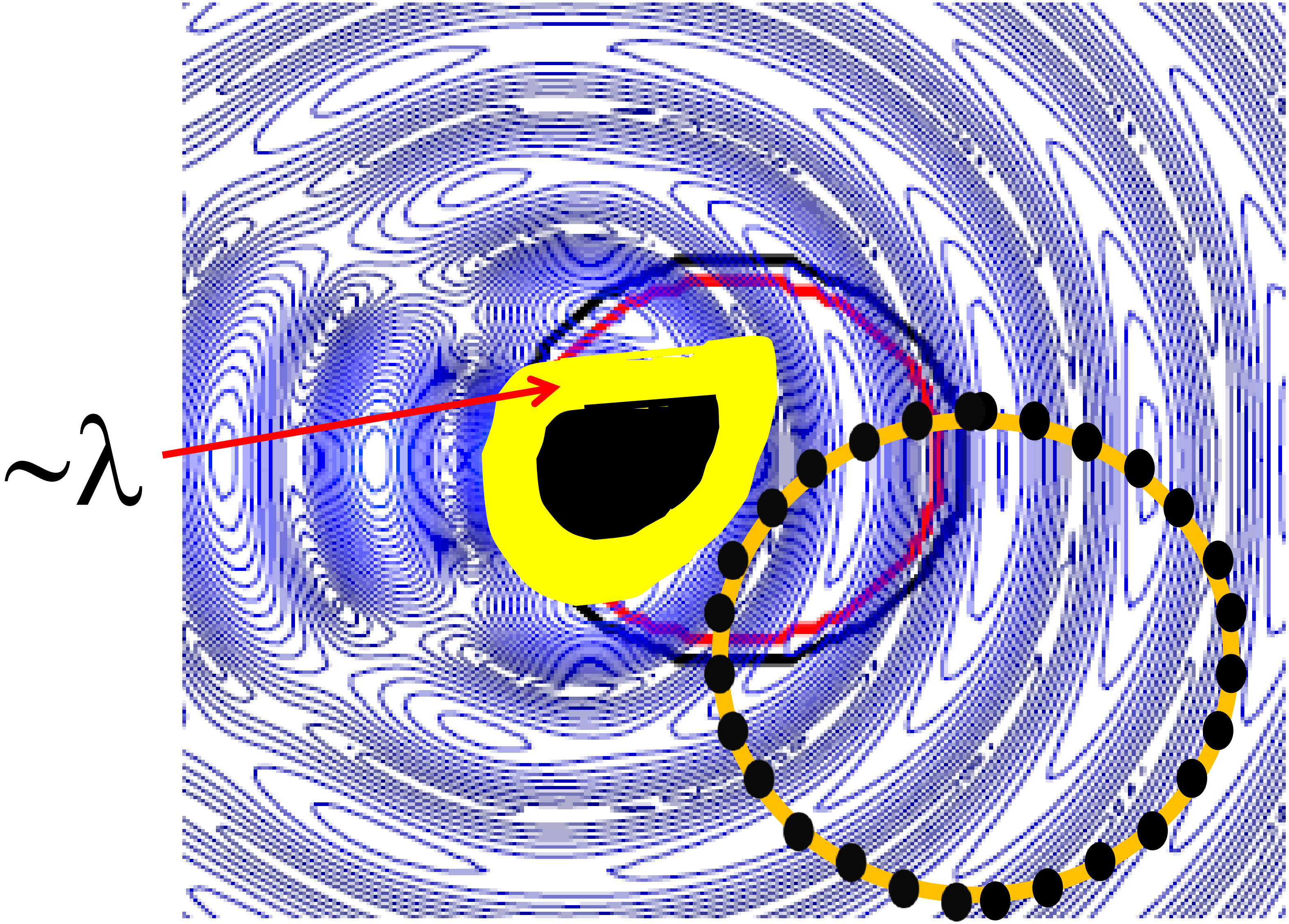


Uncertainty Region

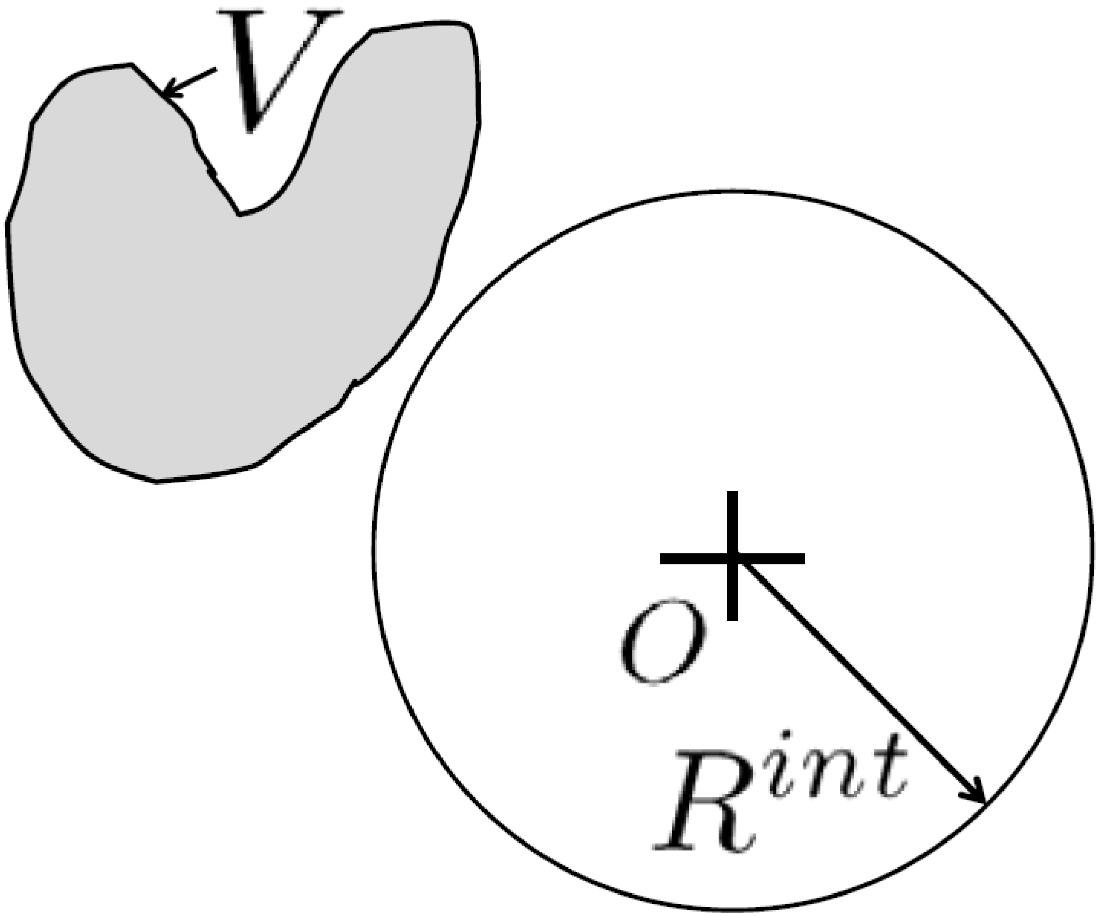
$\sim\lambda$



"Interior Problem" Backpropagation



Nonconvex Part

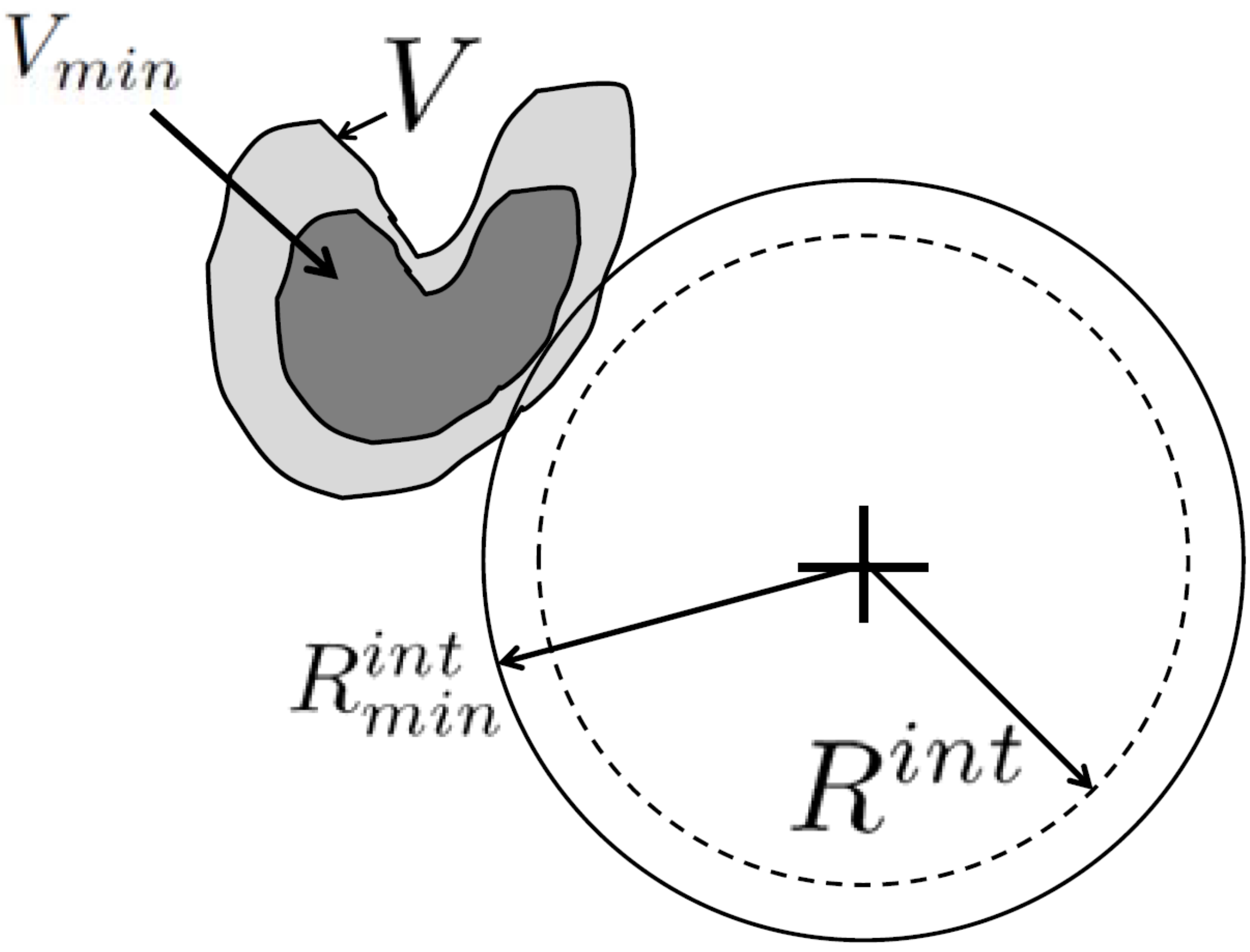


(Source-free) Multipole Expansion

$$\psi(\mathbf{r}) = ik \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{l,m} j_l(kr) Y_{l,m}(\hat{\mathbf{r}}), \quad r < R^{int}$$

$$g_{l,m} = \int_{R^{int}}^{\infty} dr r^2 h_l^{(1)}(kr) \int_{S^2} d\hat{\mathbf{r}} Y^*_{l,m}(\hat{\mathbf{r}}) \rho(\mathbf{r}).$$

Relations



Forward Problem

$$\psi_a(\hat{\mathbf{r}}) = \psi(a\hat{\mathbf{r}}) = \sum_{l,m} \bar{a}_{l,m} Y_{l,m}(\hat{\mathbf{r}}) \quad a < R_{min}^{int}$$

Knows field for radius a, computes field for radius b<a

$$\bar{a}_{l,m} = g_{l,m} j_l(ka)$$

$$\psi_b(\hat{\mathbf{r}}) \quad = \quad \sum_{l,m} \bar{b}_{l,m} Y_{l,m}(\hat{\mathbf{r}})$$

$$\bar{b}_{l,m} = \bar{a}_{l,m} j_l(kb) / j_l(ka)$$

Inverse Problem

Knows field for radius b<a, computes field for radius a

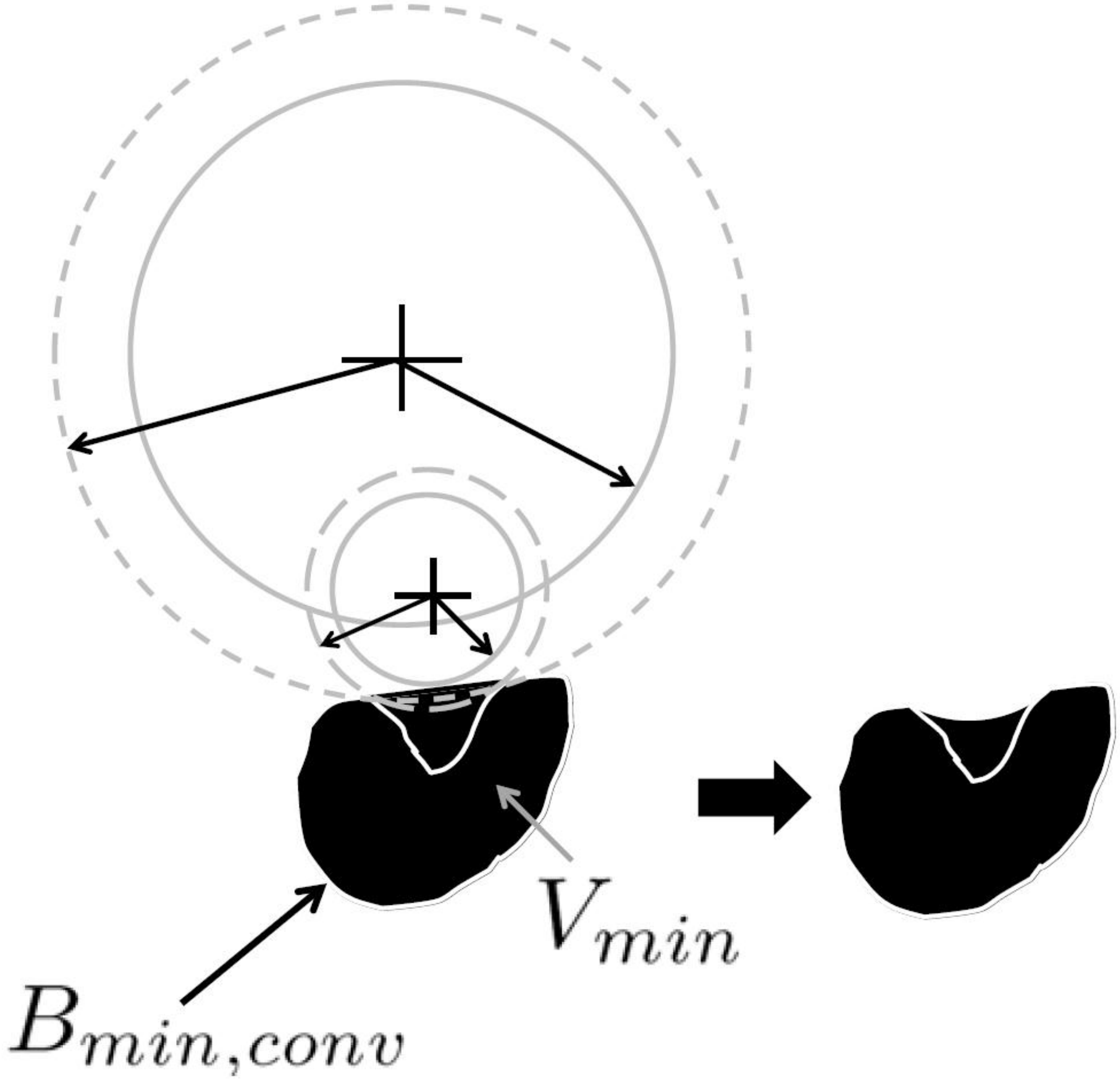
$$\psi_a(\hat{\mathbf{r}}) = \sum_{l,m} \bar{b}_{l,m} \left[\frac{j_l(ka)}{j_l(kb)}\right] Y_{l,m}(\hat{\mathbf{r}})$$

$$\sum_{l,m} |\bar{b}_{l,m}|^2 \left[\frac{j_l(ka)}{j_l(kb)}\right]^2 < \infty$$

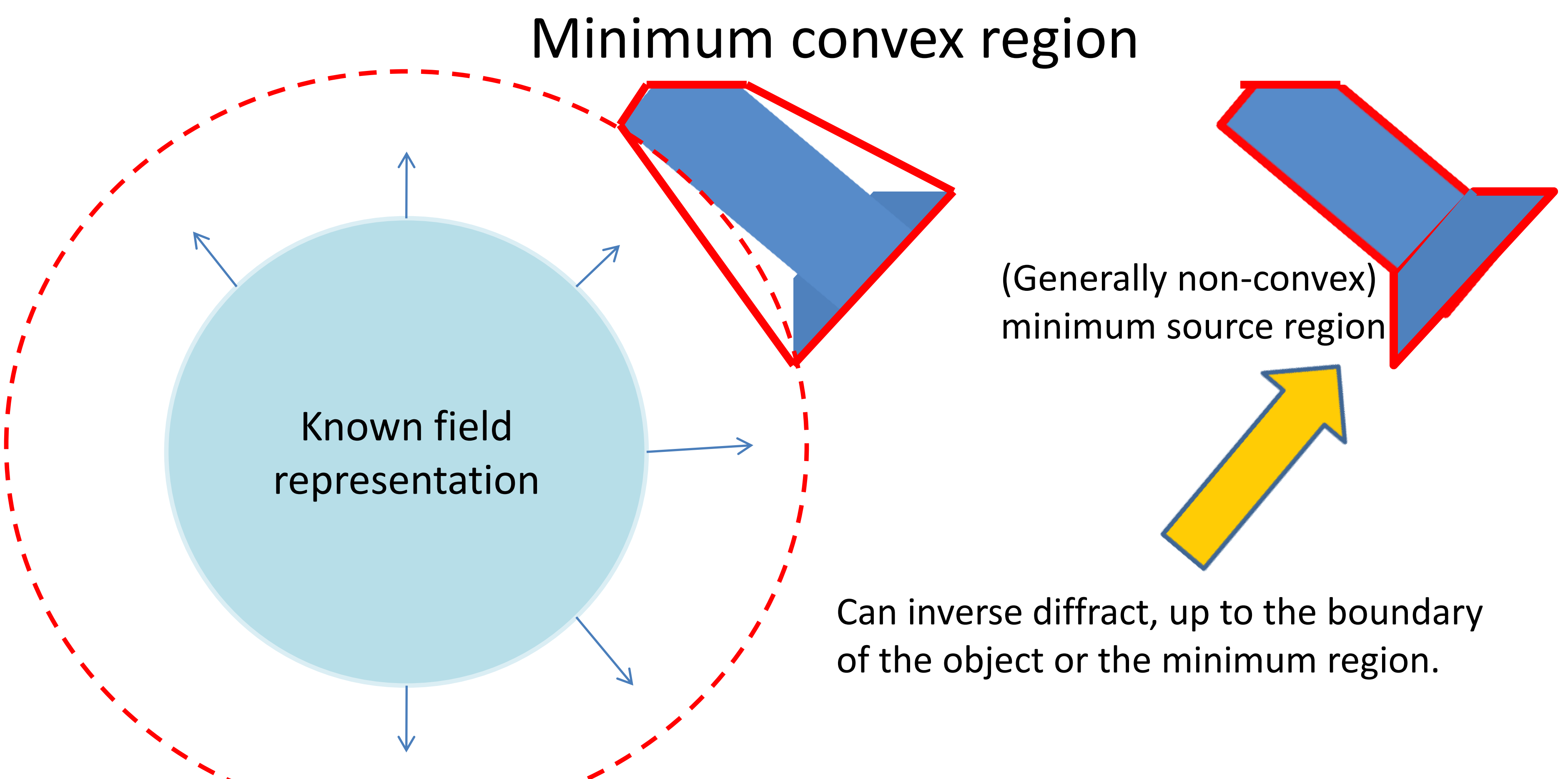
$$\lim_{l\to\infty} |\bar{b}_{l,m}|(a/b)^l \neq 0, \quad \text{at least one } m.$$

$$\boxed{R^{int}_{min} = \inf\{a \in \mathbb{R}^+ : \lim_{l\to\infty} (ka/2)^l [\Gamma(l+3/2)]^{-1} |g_{l,m}| \neq 0, \quad \text{at least one } m\}.}$$

Minimum Region Computation



Interior Problem



Analytical Results

- Point sources

- Radiating disk

- Radiating ring

Other related computations (plane wave theory):

- Parallelpiped endfire-like continuous sources
- Wavelet fields (Kaiser, with Tony Devaney (IEEE Trans.))
- Minimum support coincides with true support.

2D Formulas

Exterior

$$E_z(a,\phi) = -\frac{\omega\mu}{4}\sum_{m=-\infty}^{\infty} a_m H_m^{(1)}(ka)e^{-im\phi}$$

$$\boxed{R_{min} = \sup\{b \in \mathbb{R}^+ : \lim_{m\to\infty}\Gamma(m)\left(\frac{2}{kb}\right)^m |a_m| \neq 0\}.}$$

Interior

$$E_z(b,\phi) = -\frac{\omega\mu}{4}\sum_{m=-\infty}^{\infty} g_m J_m(kb)e^{-im\phi}$$

$$\boxed{R_{min}^{int} = \inf\{a \in \mathbb{R}^+ : \lim_{m\to\infty}\left(\frac{ka}{2}\right)^m |g_m|/\Gamma(m+1) \neq 0\}}$$

Computational Implementation

Exterior

$$E_z(a,\phi) \doteq -\frac{\omega\mu}{4} \sum_{m=-kR}^{kR} a_m H_m^{(1)}(ka) e^{-im\phi}$$

$$R_{min} = \inf\{b \in \mathbb{R}^+ : \sum_{m=-kR}^{kR} |a_m|^2 |H_m^{(1)}(kb)|^2 < B\}$$

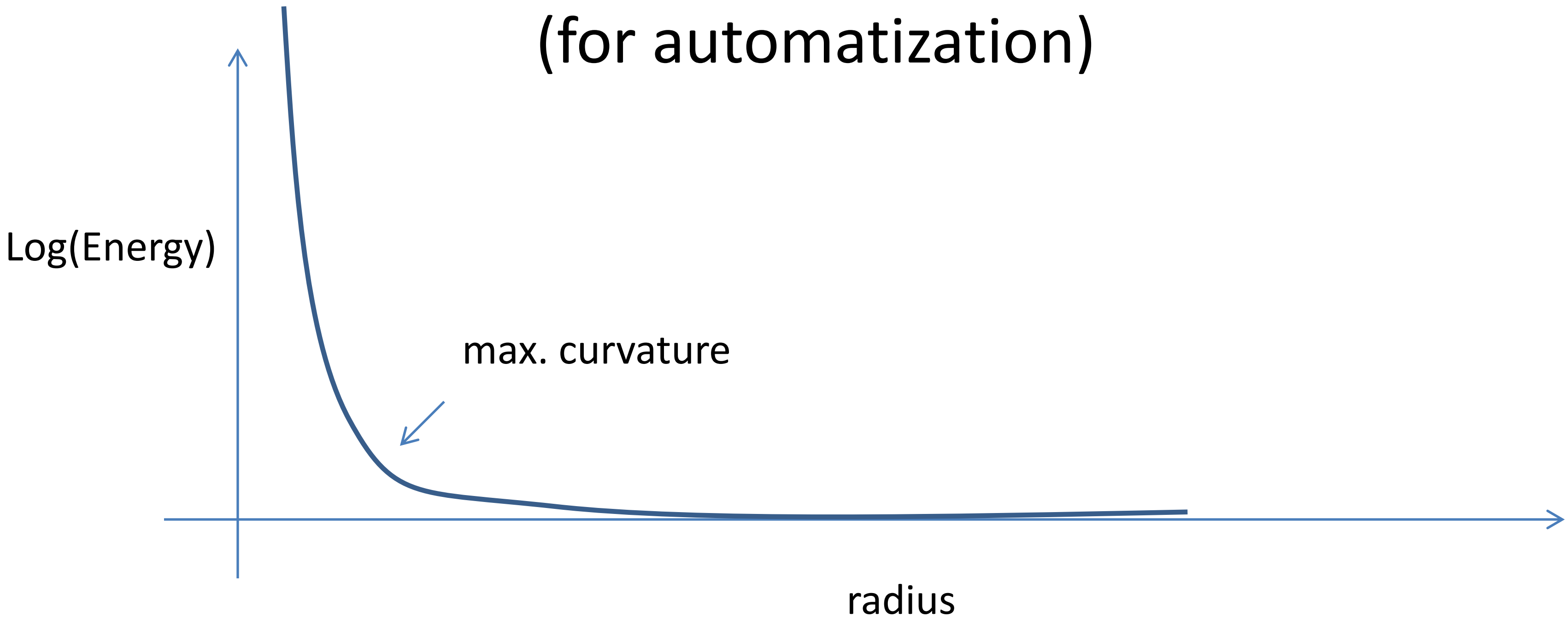
Interior

$$E_z(b,\phi) \doteq -\frac{\omega\mu}{4} \sum_{m=-kR}^{kR} g_m J_m(kb) e^{-im\phi}$$

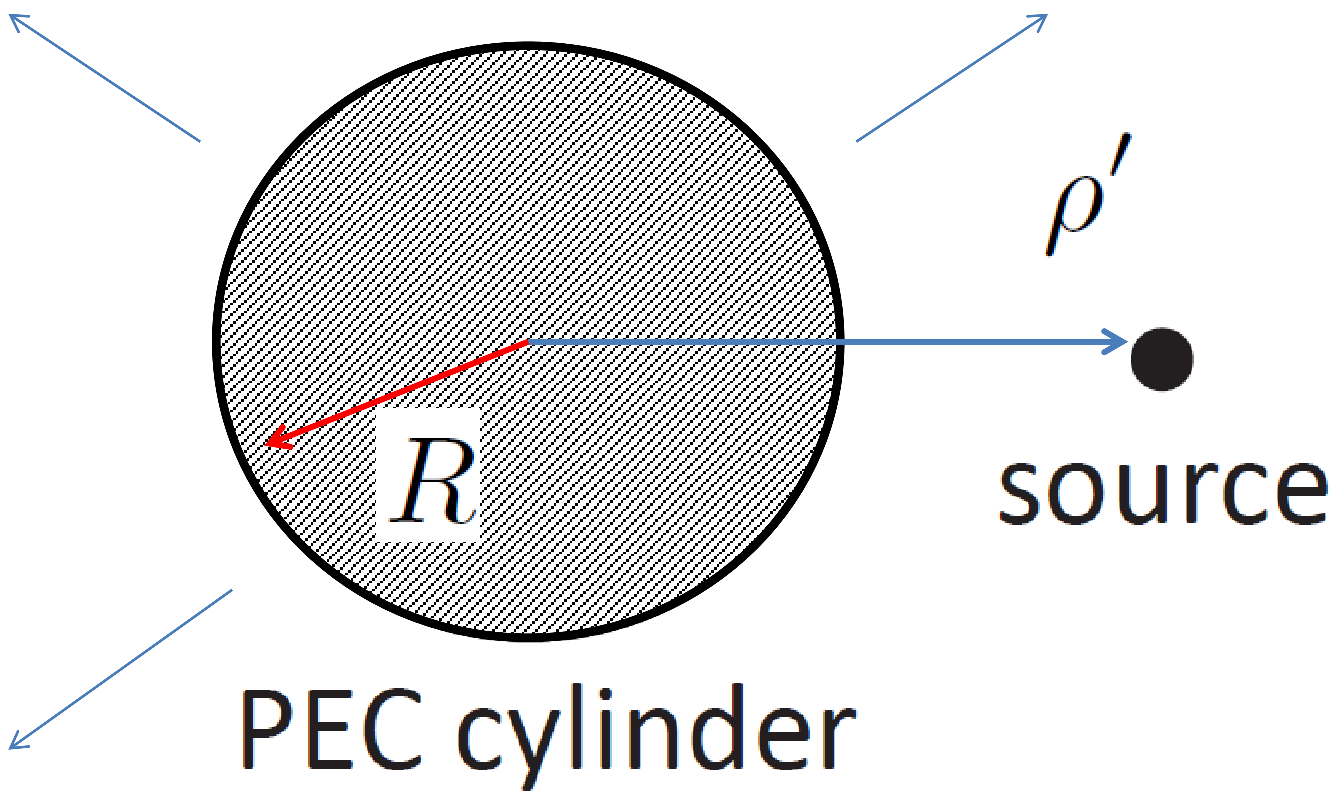
$$R_{min}^{int} = \sup\{a \in \mathbb{R}^+ : \sum_{m=-kR}^{kR} |g_m|^2 |J_m(ka)|^2 < B\}$$

Cutoff

(for automatization)



Example: (2D) Scattering by Cylinder



Example: (2D) Scattering by Cylinder

$$E_z(\rho, \phi) = -\frac{k^2 I_e}{4\omega\epsilon} \sum_{m=-\infty}^{\infty} a_m H_m^{(1)}(k\rho) e^{-im\phi} \quad \rho > R$$

$$a_m = -\frac{J_m(kR)}{H_m^{(1)}(kR)} H_m^{(1)}(k\rho') e^{im\phi'}$$

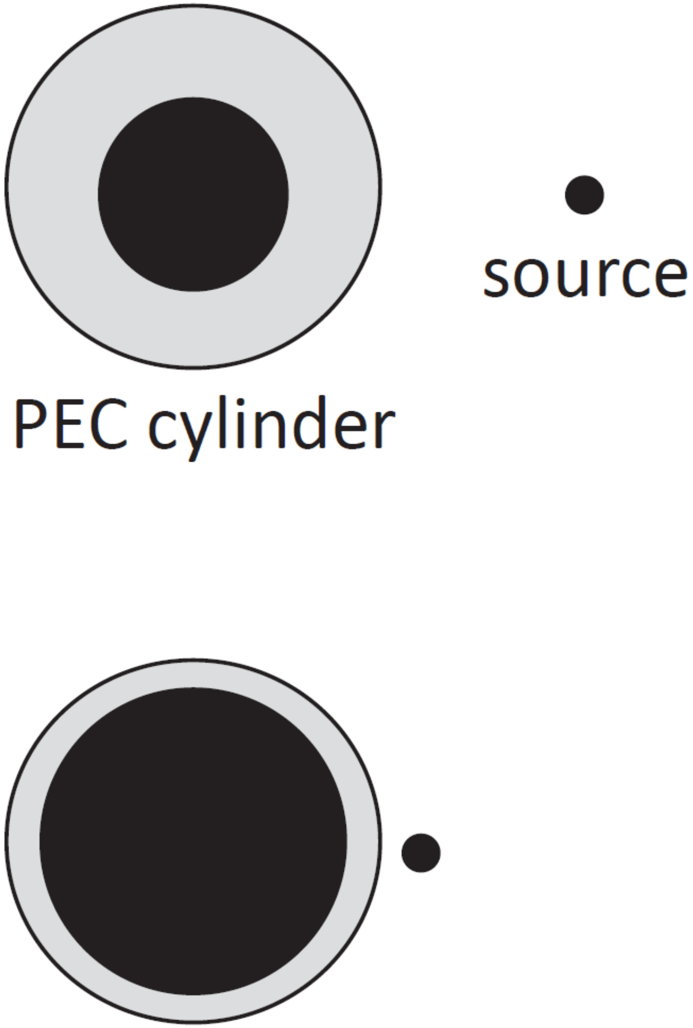
$$\lim_{m\to\infty} \Gamma(m) \left(\frac{2}{kb}\right)^m |a_m| \quad = \quad 0 \quad b > R^2/\rho'$$

Noninfinite, yet
may be a large number

$$\lim_{m\to\infty} \Gamma(m) \left(\frac{2}{kb}\right)^m |a_m| \quad = \quad \infty \quad b < R^2/\rho'.$$

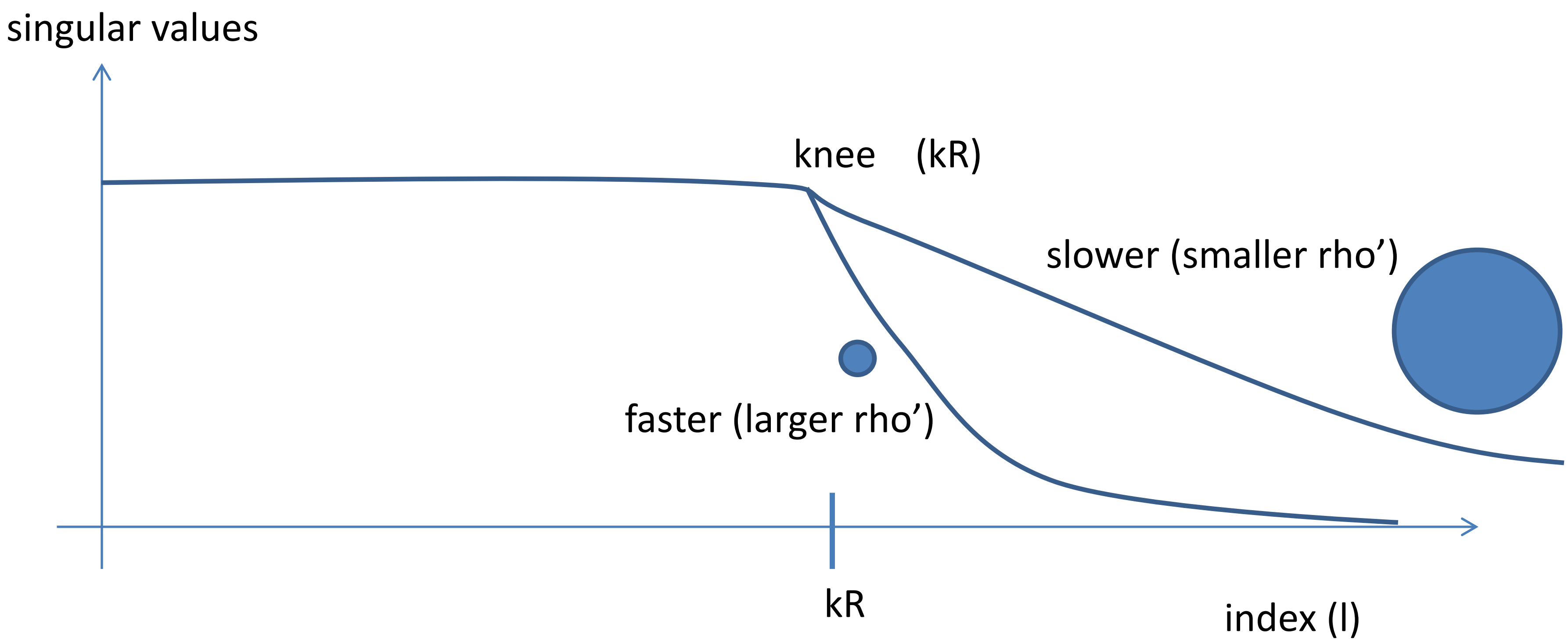
$$B_{min} = \{\mathbf{r} \in \mathbb{R}^2 : \rho \leq R^2/\rho'\}$$

Example of the UWSC Sets



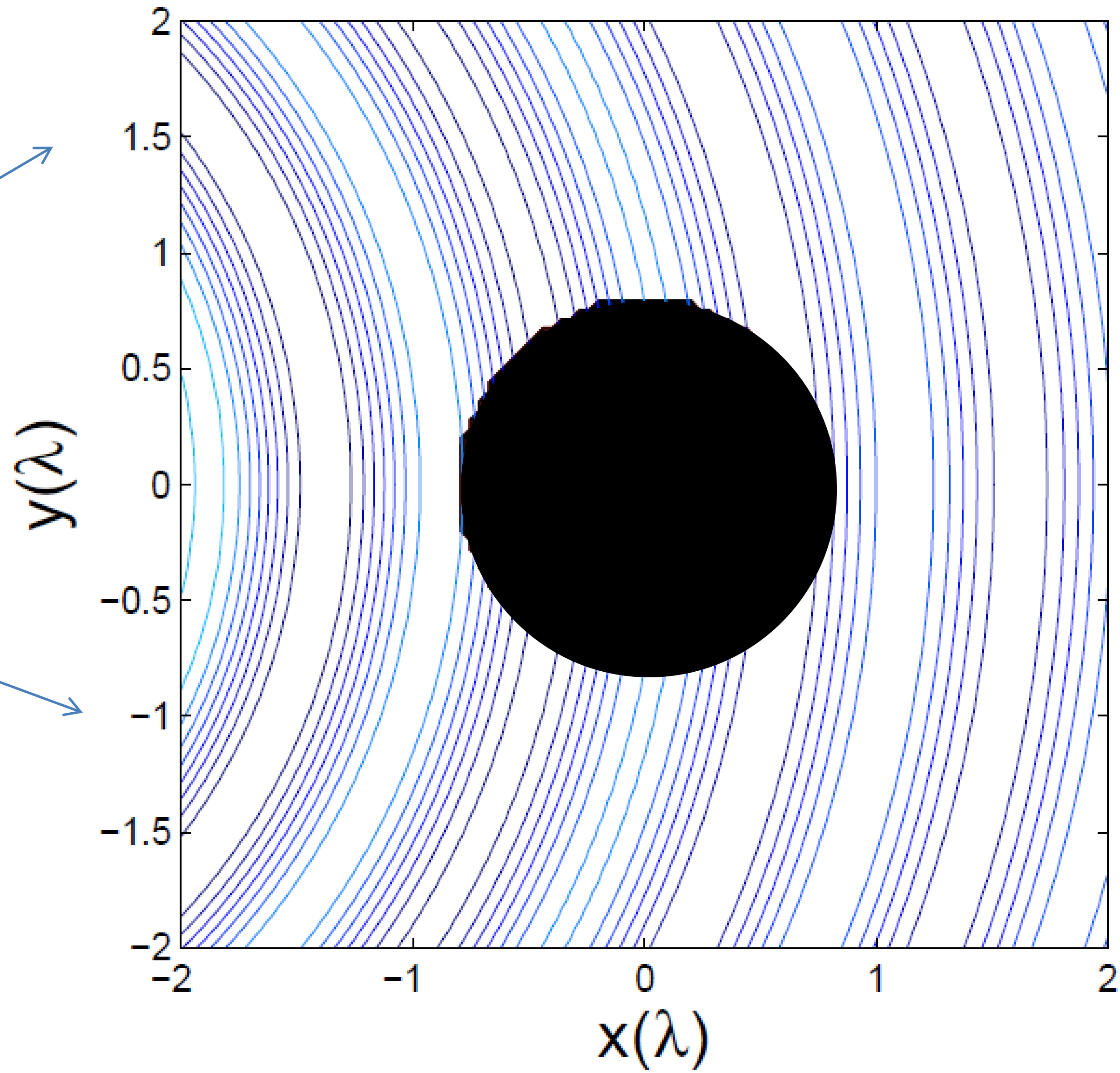
Noninfinite, yet
may be a large number

Essential NDF



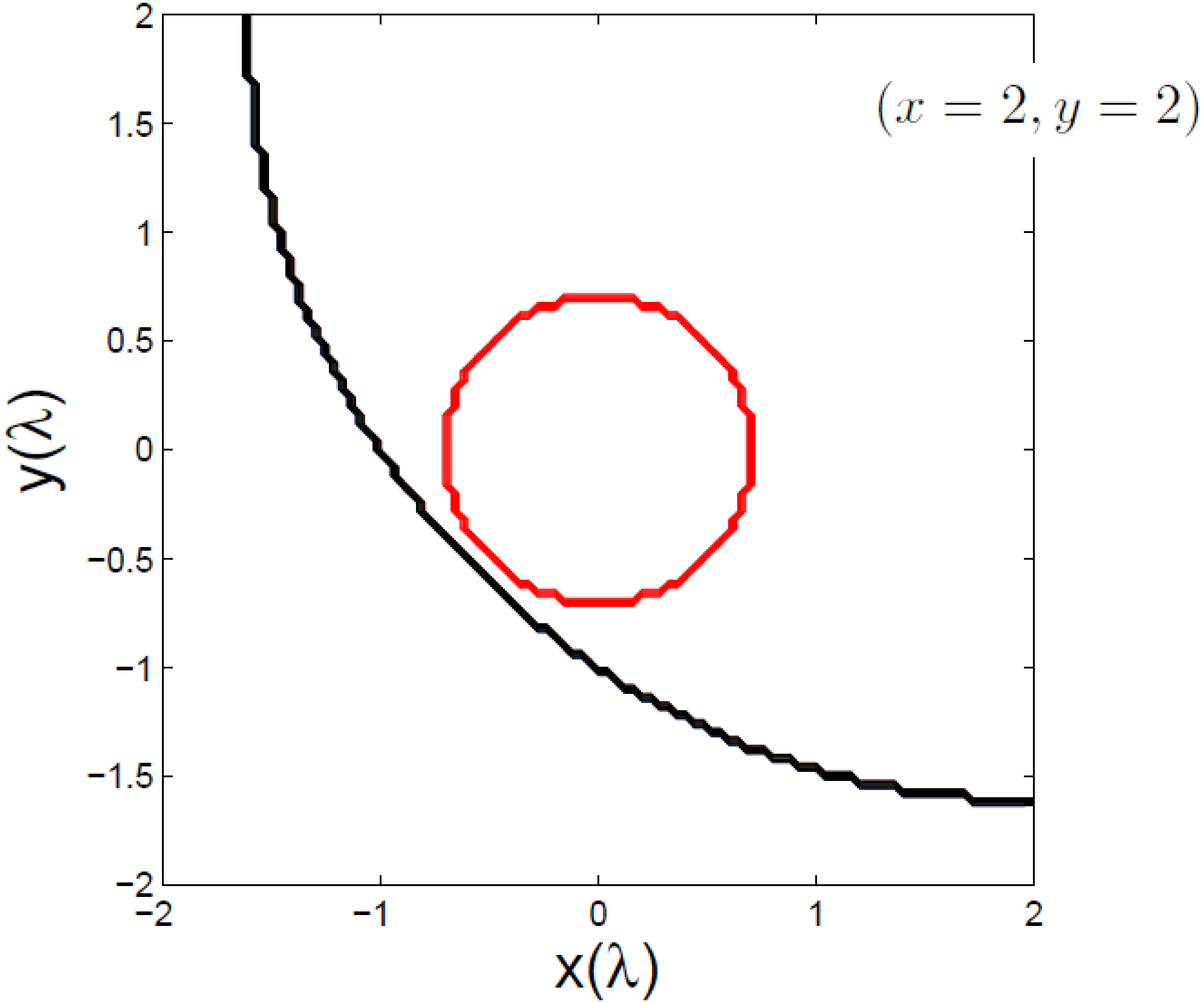
Example

Point source at (-4,0)

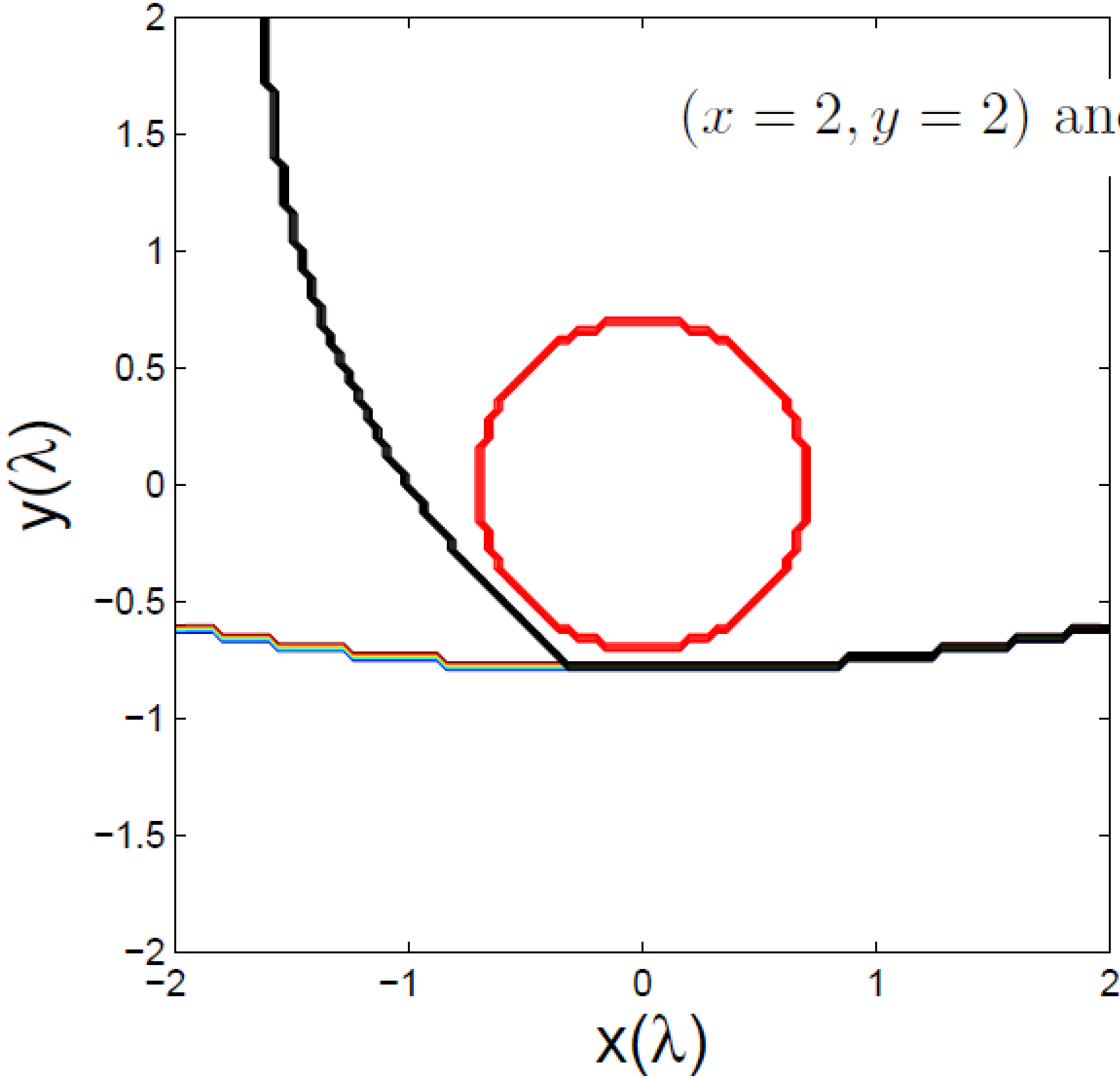


PEC cylinder

Example

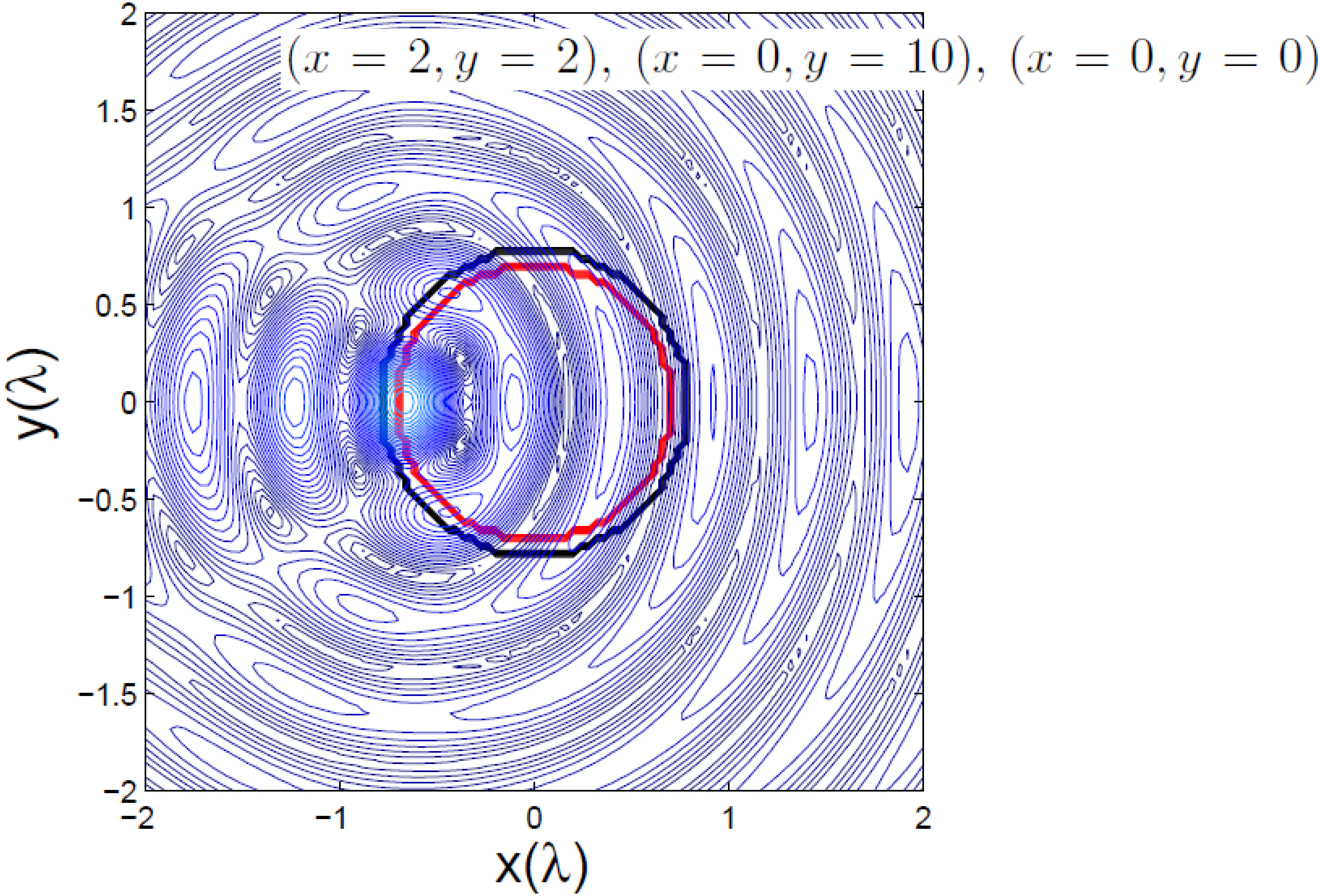


Example

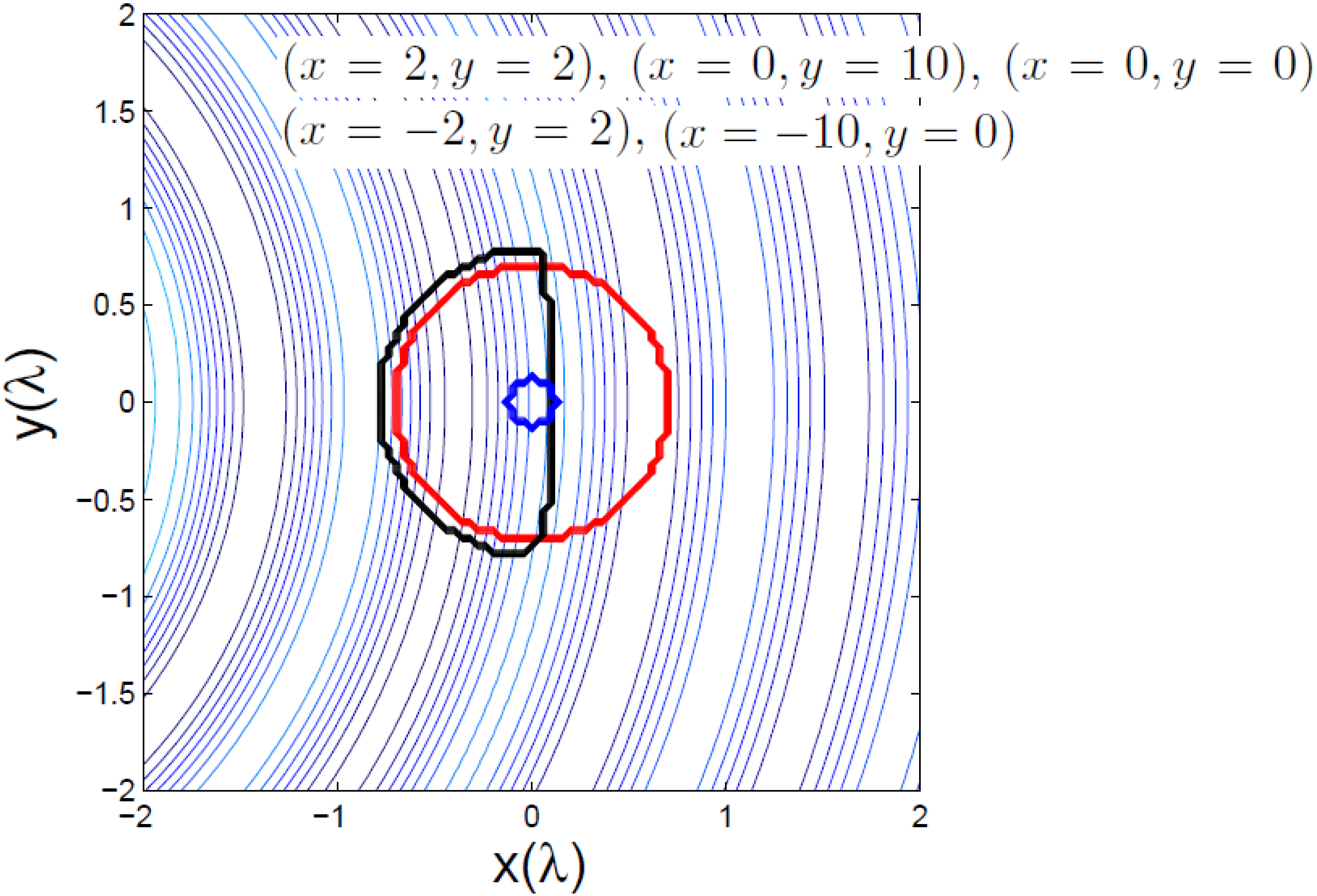


$(x = 2, y = 2)$ and $(x = 0, y = 10)$

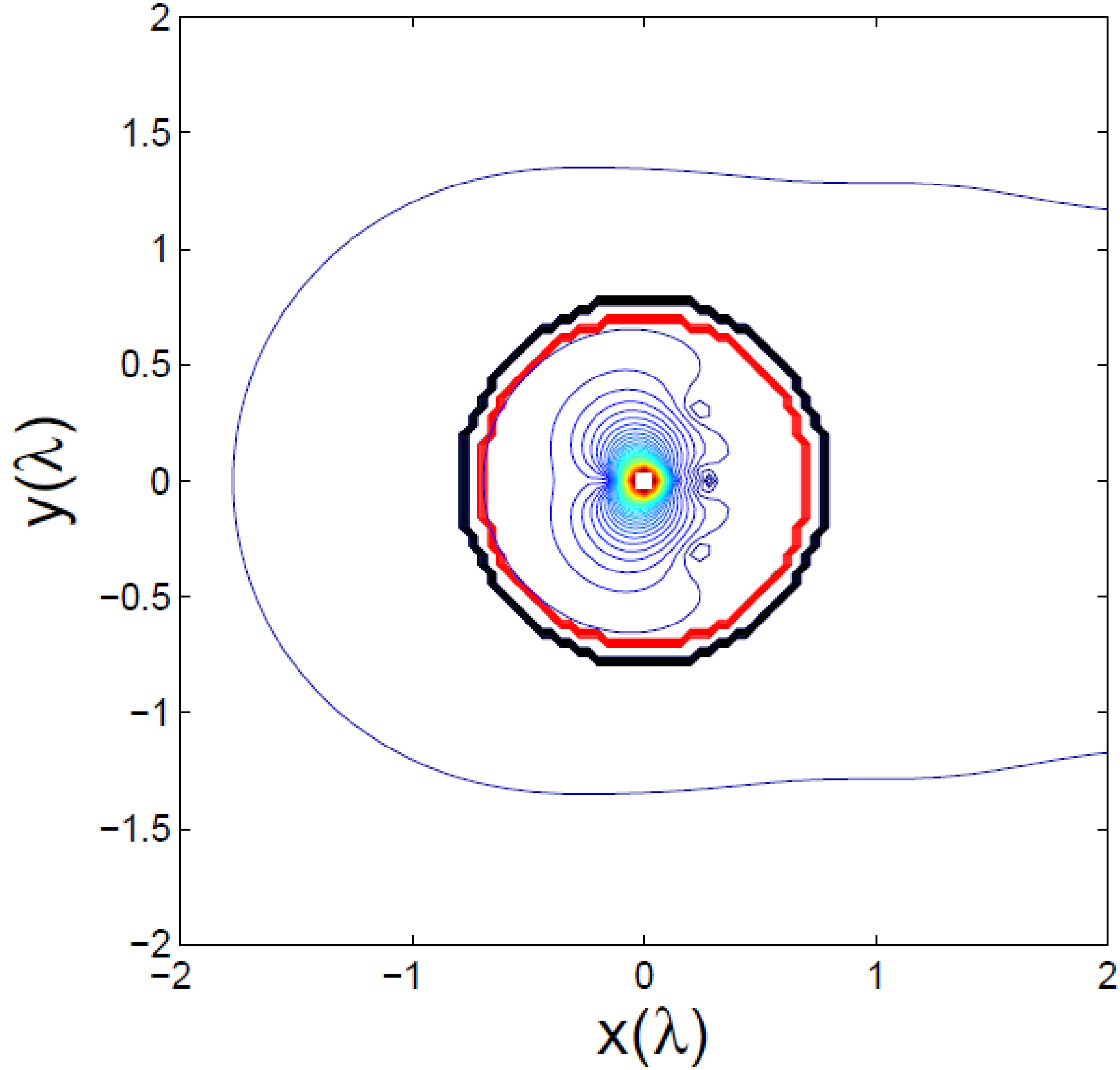
Example



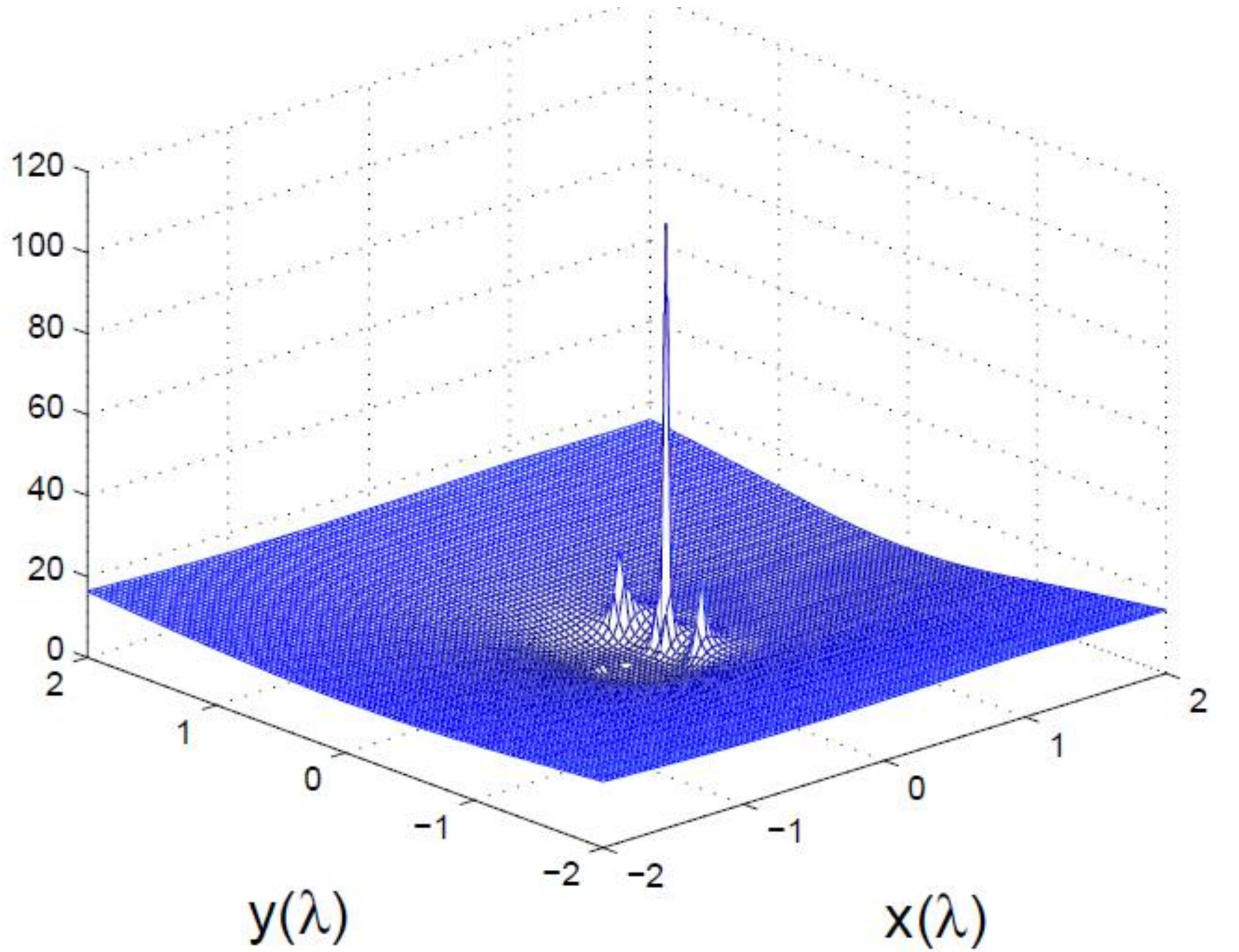
Example



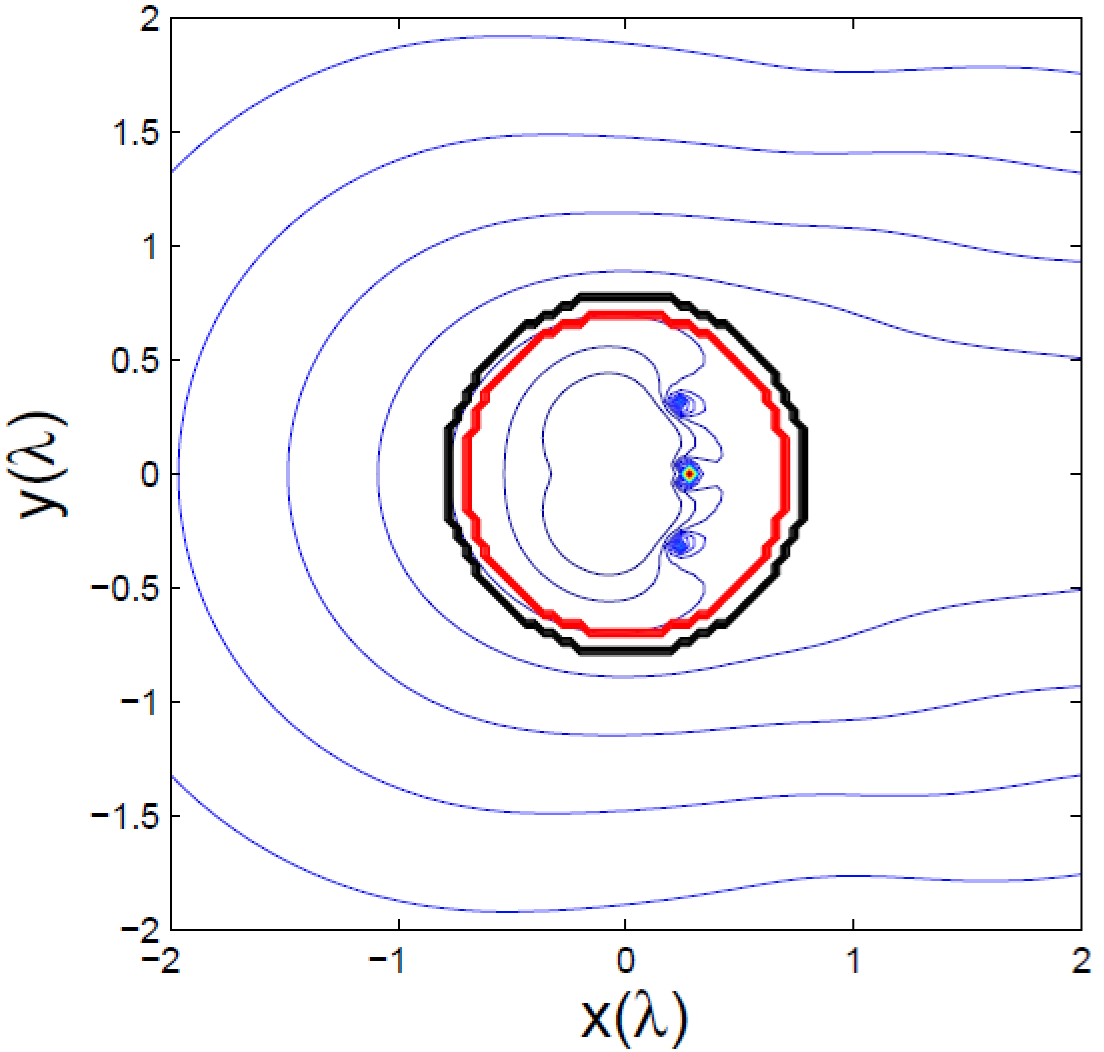
Hankel-function-based Backpropagation



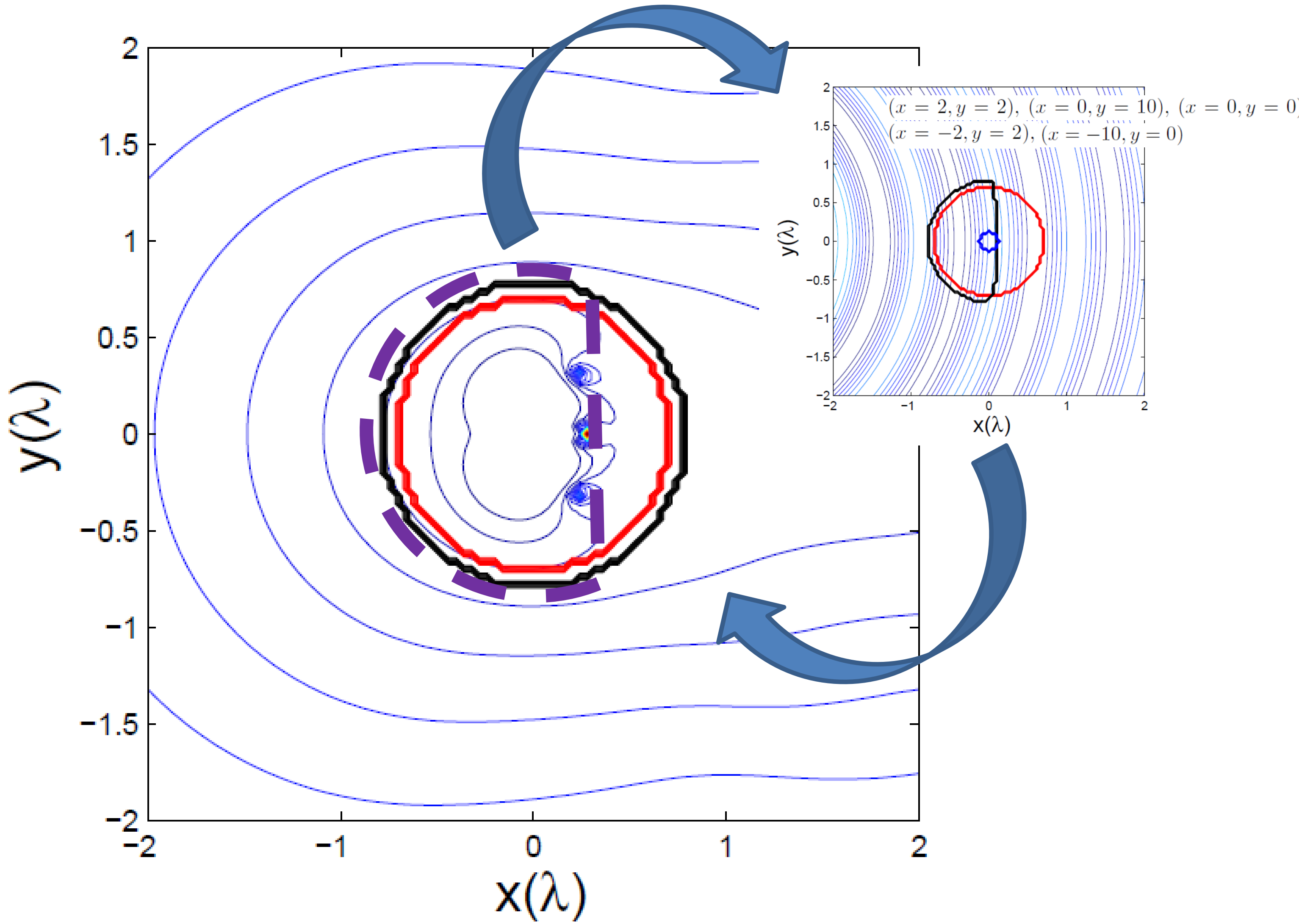
Inverse of the Hankel-function-based Backpropagation Pseudospectrum



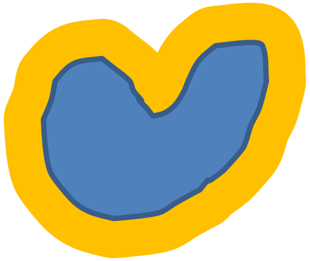
Inverse of the Hankel-function-based Backpropagation Pseudospectrum



Inverse of the Hankel-function-based Backpropagation Pseudospectrum

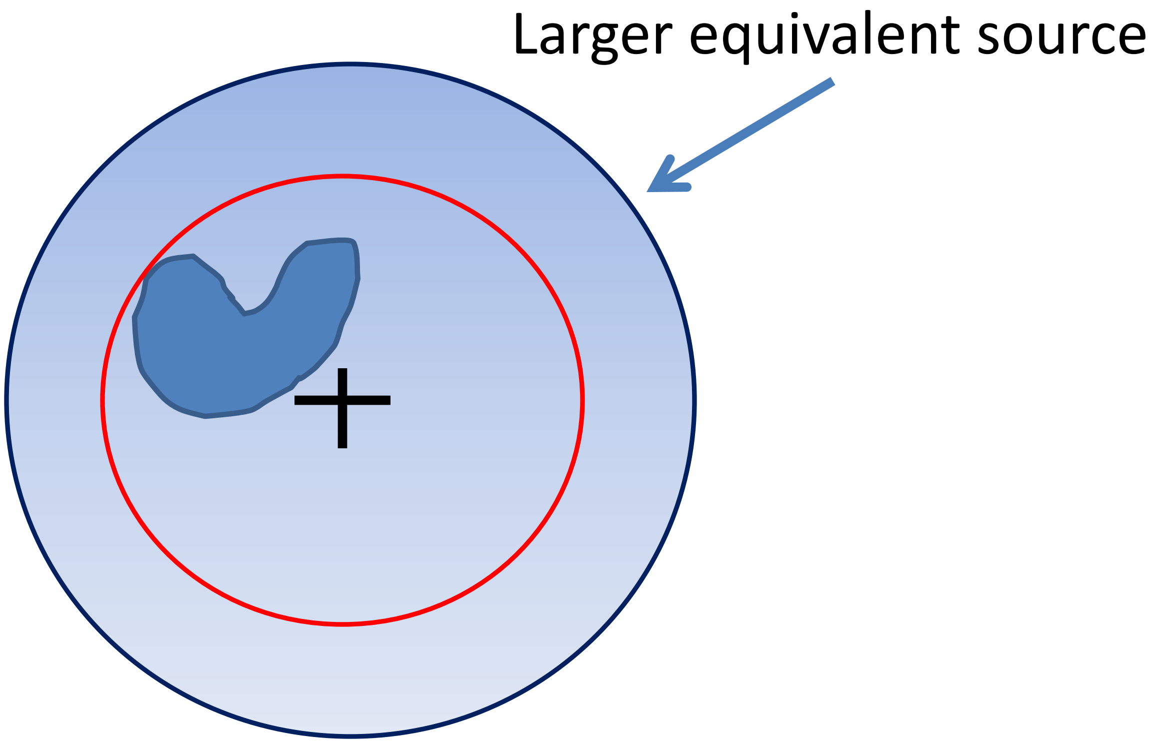


Finite-dimensional Approximation and Probabilistic Insight



Original source (orange), and the minimum source region of its far field (blue)

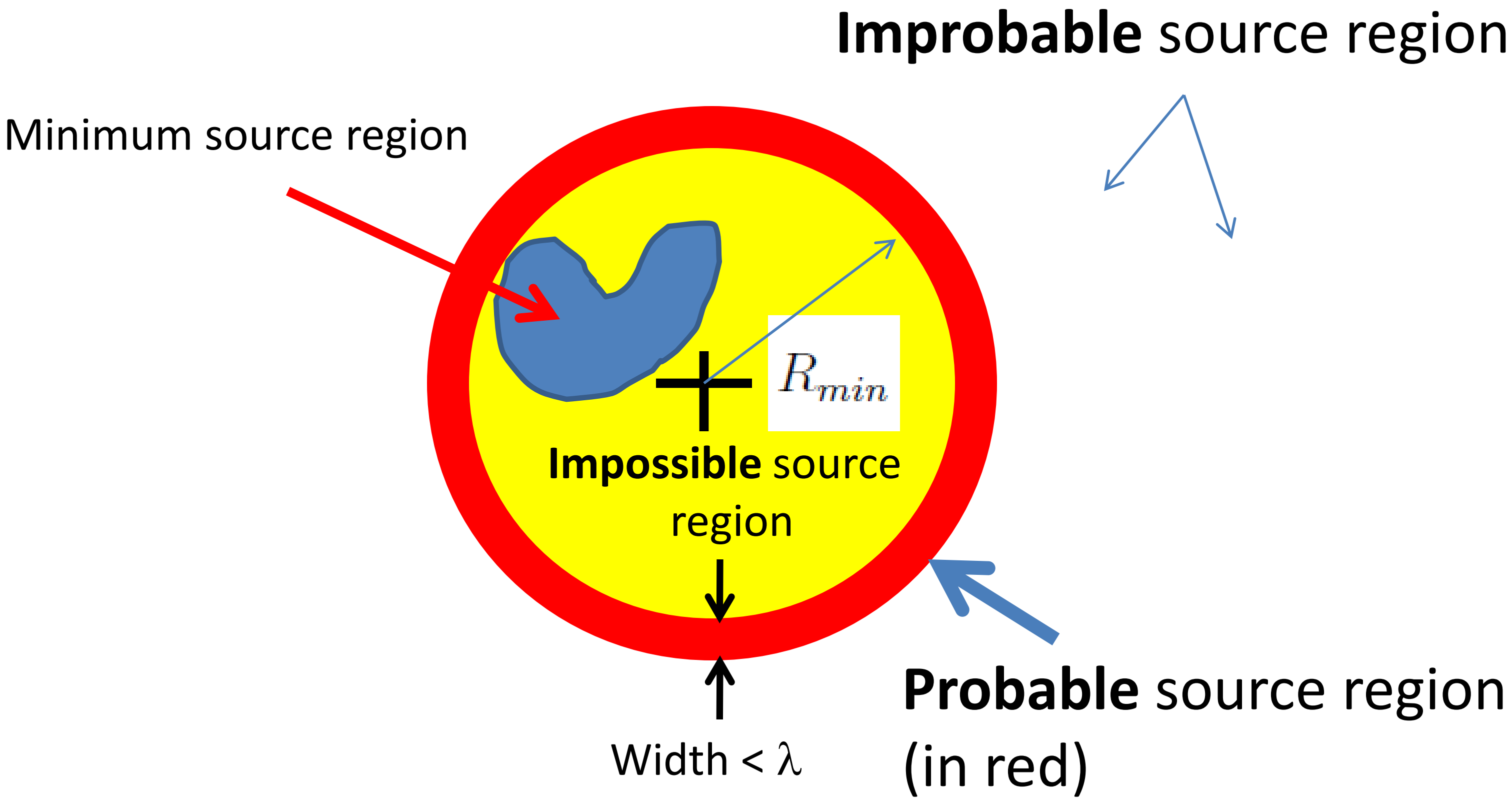
Nonuniqueness



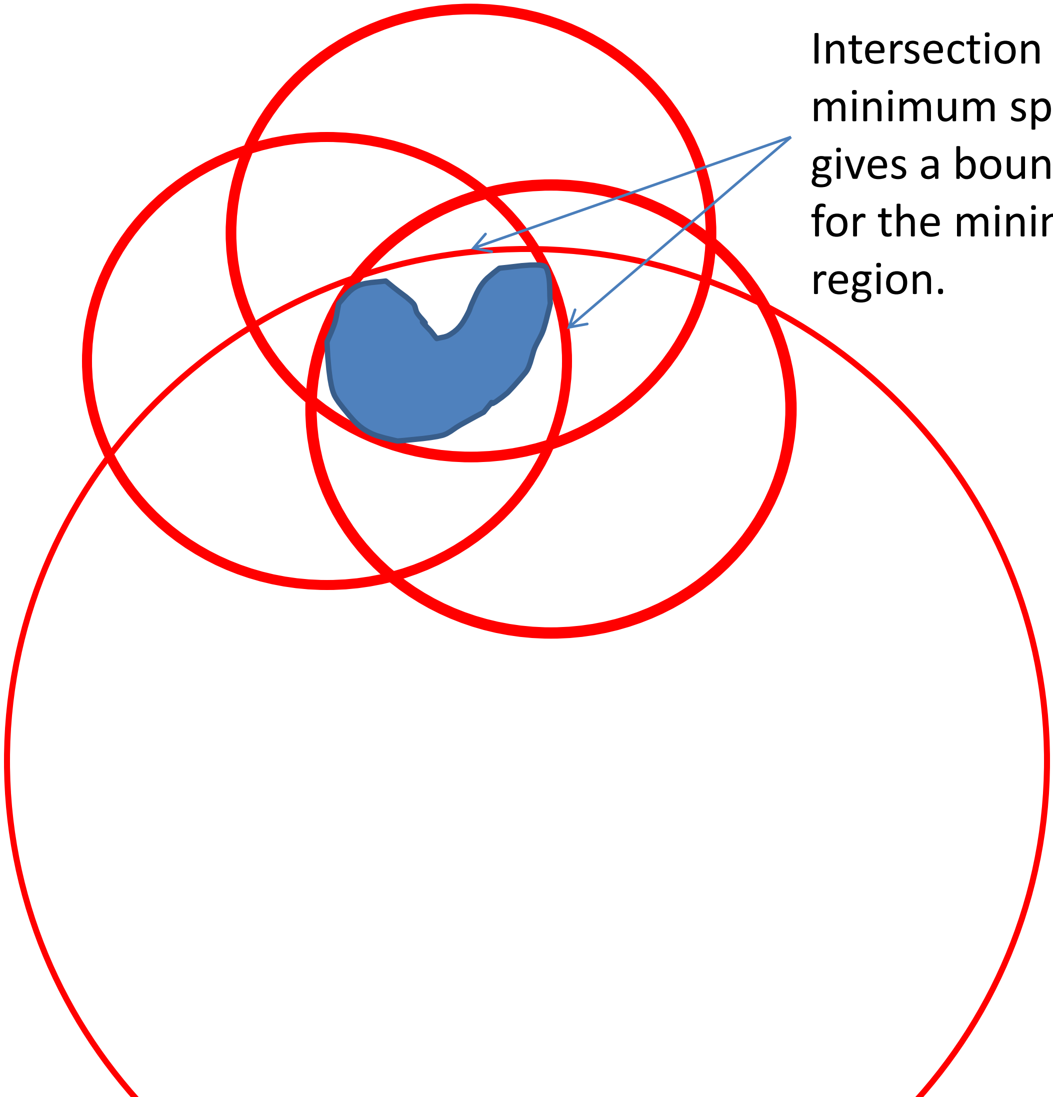
There is an **infinite number** of equivalent sources larger than the minimum volume.

In a real-world imaging problem, how useful is it to know the minimum region?

Confidence Interval



Multiple Origins



Intersection of the minimum spherical volumes gives a bounding region for the minimum source region.

Essentially finite-dimensional Space

If $R_{min} = R$ then $a_{l,m} \simeq 0$ for $l \gtrsim ka \simeq kR$ (where $m = -l, -l + 1, \cdots, l - 1, l$).

If, on the other hand, $k(R - R_{min}) \simeq 1$, meaning $\boxed{R_{min} \simeq R - \lambda/2\pi,}$ then the source ρ would have to be such that

$$a_{l,m} = \int_{r \leq R} dr r^2 j_l(kr) \int_{S^2} d\hat{\mathbf{r}} Y^*_{l,m}(\hat{\mathbf{r}}) \rho(\mathbf{r}) \simeq 0$$

for $l \simeq kR$, $m = -kR, -kR + 1, \cdots, kR - 1, kR$, which is a specialized condition.

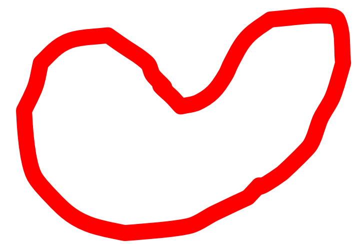
Confidence Intervals

Thus for the given R the entire signal space is $S_R = \{a_{l,m}, l = 0, 1, \cdots, \simeq kR, m = -l, -l+1, \cdots, l-1, l\}$, corresponding to $(kR+1)^2$ dimensions. In contrast, if $R_{min} \simeq R - \lambda/2\pi$ the respective signal space is $S_{R-\lambda/2\pi} = \{a_{l,m}, l = 0, 1, \cdots, \simeq k(R-1), m = -l, -l+1, \cdots, l-1, l\}$ which means a drastic dimensionality contraction of $2kR+1$ dimensions, in a space of $(kR+1)^2$ dimensions, which is very unlikely.

If the far field data vectors $(a_{l,m})$ are randomly picked they will land with zero probability in the subspace $S_{R-\lambda/2\pi} \subset S_R$.

Tighter Bound

$\sim \lambda/2\pi$



Furthermore, the estimate

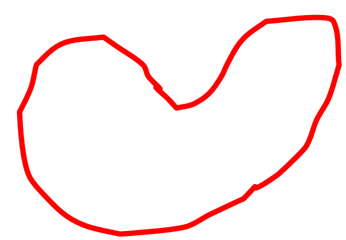
$$R - \lambda/2\pi \lesssim R_{min} \leq R$$

is very conservative.

For the given R, $NDF = (kR+1)^2$. Similarly, for the radius R_{min} associated to the given far field, $NDF = (kR_{min}+1)^2$. Then for a dimensionality reduction of only 1 we have

$$(kR+1)^2 \geq (kR_{min}+1)^2 + 1$$

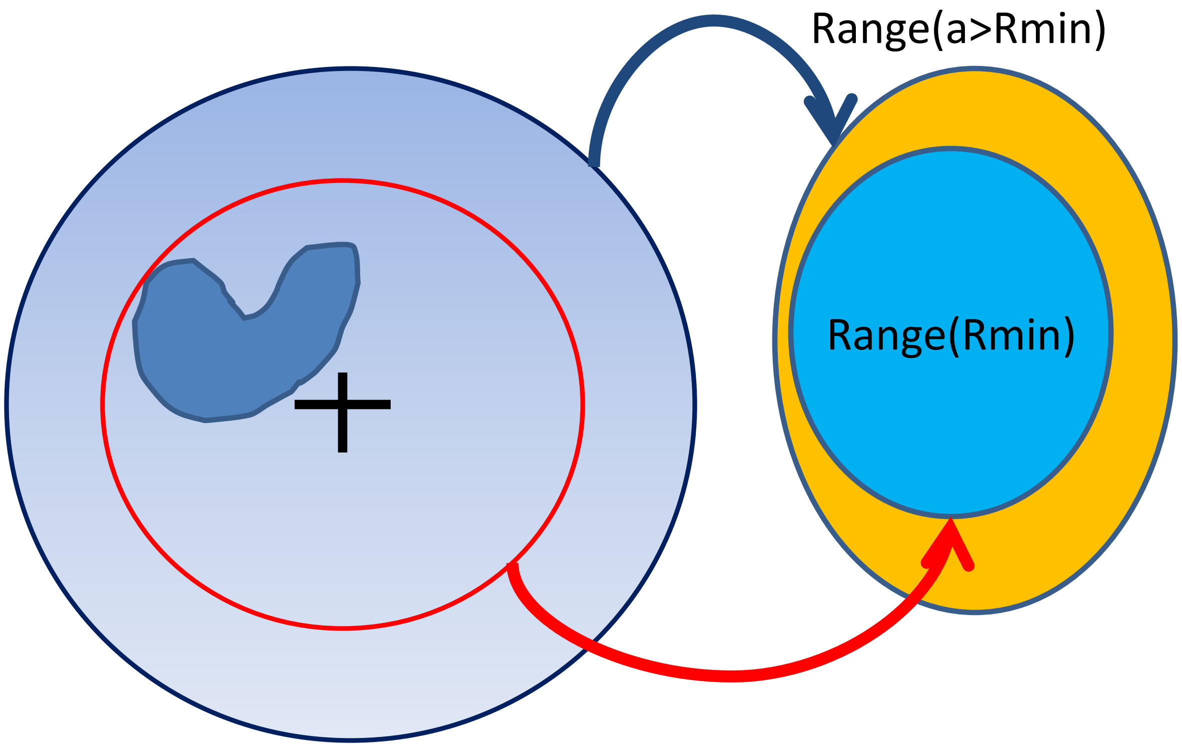
which gives the tighter bound



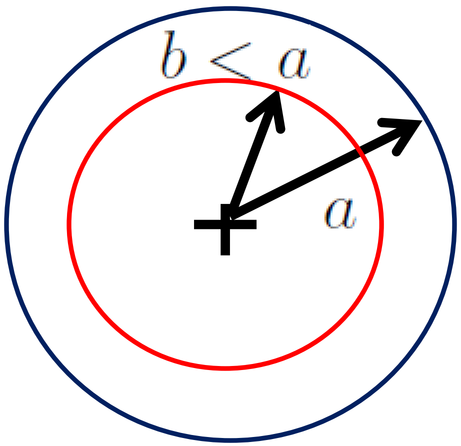
$$R - \left(\frac{\lambda}{2\pi}\right)^2 \frac{1}{2R} \lesssim R_{min} \leq R.$$

$<< \lambda$

Analysis in infinite-dimensional Space



Ranges



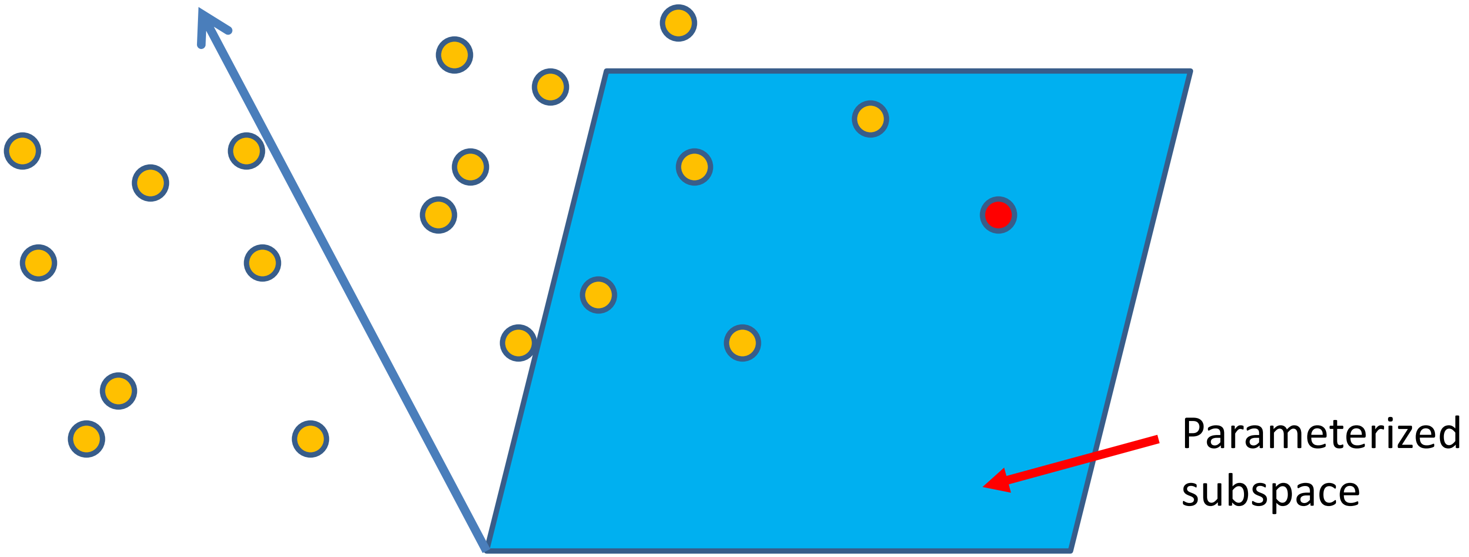
$$\mathrm{Range}(a) : \sum_{l,m} |a_{l,m}|^2/\sigma_l^2(a) < \infty$$

$$\mathrm{Range}(b < a) : \sum_{l,m} |a_{l,m}|^2/\sigma_l^2(b) < \infty \quad b < a$$

Since $\sigma_l(a) > \sigma_l(b)$ for $a > b$ then $\mathrm{Range}(b < a) \subset \mathrm{Range}(a)$

The key question: How large should a be (relative to given b representing the minimum region) such that the complement of $\mathrm{Range}(b)$ in $\mathrm{Range}(a)$ is non-empty? (has essential dimension)

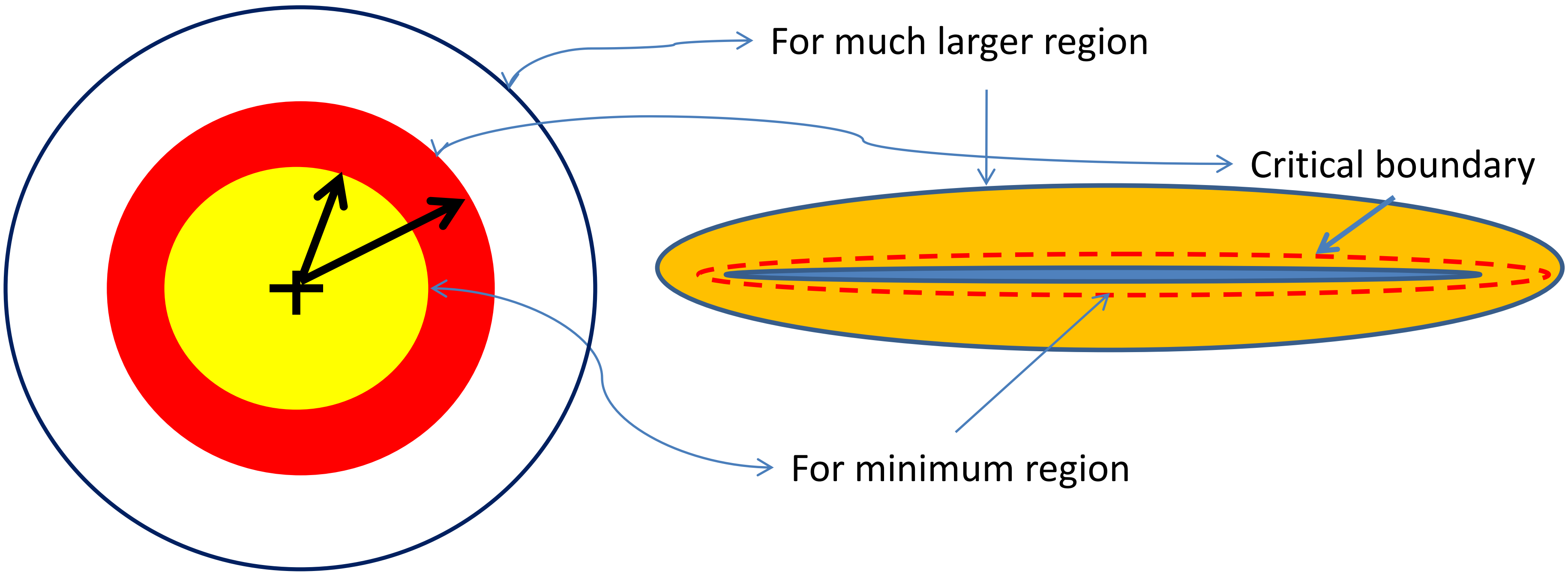
Unlikely Dimensionality Reduction



Random population of a space

Can be thought of as a matching of data and source complexities (sparsity).

Far Field Manifolds



Source (configuration) space

Field multipole domain

(angular momentum)

Bounded Source Energy

Consider the related bounded energy constraint

$$\mathrm{Range}(a;B):\sum_{l,m}|a_{l,m}|^2/\sigma_l^2(a)<B<\infty$$

$$\mathrm{Range}(b<a;B):\sum_{l,m}|a_{l,m}|^2/\sigma_l^2(b)<B<\infty\quad b<a$$

n-ellipsoids

Consider the shorthand notation $(l, m) \to j$.

For $j = 1, 2, \cdots, n$, we have the n-ellipsoid in far-field space, corresponding to source radius a

$$\mathrm{Man}(a; B, n) : \sum_{j=1}^{n} |a_j|^2 / [B\sigma_j^2(a)] < 1$$

whose volume is

$$V(a; B, n) = \sqrt{B}\frac{\pi^{n/2}}{\Gamma(n/2+1)}\Pi_{j=1}^{n}\sigma_j(a)$$

n-ellipsoids (cont.)

Similarly, for the smaller source radius $b < a$

$$\mathrm{Man}(b; B, n) : \sum_{j=1}^{n} |a_j|^2 / [B\sigma_j^2(b)] < 1$$

so that

$$V(b; B, n) = \sqrt{B} \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \Pi_{j=1}^{n} \sigma_j(b)$$

Probability

Therefore, the probability that a randomly picked multipole vector $\{a_{l,m}\}$ in the n-dimensional manifold $\mathrm{Man}(a; B, n)$ of B-bounded-energy sources confined within the ball of radius a lies in the (smaller) manifold $\mathrm{Man}(b; B, n)$ corresponding to B-bounded energy sources confined within the ball of radius $b < a$ is

$$P(a, b; n) = \Pi_{j=1}^{n} P_j(a, b)$$

where

$$P_j(a, b) = \frac{\sigma_j(b)}{\sigma_j(a)}$$

Probability (cont.)

Returning back to the original notation,

$$P(a, b; l_{max}) = \Pi_{l=1}^{l_{max}} [P_l(a, b)]^{2l+1}$$

where

$$P_l(a, b) = \frac{\sigma_l(b)}{\sigma_l(a)}$$

For $b < a$ we have $\sigma_l(b) < \sigma_l(a)$, so that $P_l(a, b) < 1$.

Probability (cont.)

For large l, we use the asymptotics

$$P_l(a,b) \sim \left(\frac{b}{a}\right)^{2l+3}$$

hence

$$\lim_{l\to\infty} P_l(a,b) = 0$$

and for large l_{max} the probability $P(a,b;l_{max}) = 0$.

Detailed Bounds

Theorem.

Consider sufficiently small α which allows for a sufficiently large l_0 such that $\sigma_{l_0}(b) = \alpha$, then $P_l^2(a, b; \alpha) = \frac{2}{1+\sigma_l^2(a)/\sigma_l^2(b)}$.

Set $[P_{l_0}(a, b, \alpha)]^{2l_0+1} < \epsilon < 1$.

We know from $P_l(a, b; \alpha) < 1$ that

Generally you solve this computationally

$$P(a, b; \alpha) < P_{l_0}^{2l_0+1}(a, b; \alpha) = \left[\frac{2}{1 + \sigma_l^2(a)/\sigma_l^2(b)}\right]^{2l_0+1} < \epsilon$$

therefore using the asymptotics for large l_0

$$a > b(2)^{\frac{1}{2l_0+3}} \left[\frac{1}{\epsilon^{2l_0+1} - 1/2}\right]^{\frac{1}{2l_0+3}} \sim b$$

for large l_0, as expected.

State of the Art Context

There are great shape reconstruction methods in inverse scattering: MUSIC, factorization, linear sampling, no response test, compressive imaging.

Multipole inverse support theory sheds light from the rigorous inverse diffraction point of view on the limitations and possibilities of inverting true supports from far field data.

Unlike most existing methods, this approach holds for the single transmit experiment case. This also creates a concern related to uniqueness that can be tackled via a probabilistic inverse theory, which quantifies the method’s robustness.

State of the Art Context (2)

Multipole inverse support theory has a big place:

1- rigorously quantifies information about the source localization contained in the far field.

2- “Single-probing-field” method.

3- Surprisingly robust in inverse scattering applications despite expected nonuniqueness (the bound is “tight”).

4- Gives rise to a new probabilistic inverse theory.

5- Key appeal: Universality of the approach.

Conclusions

- Outlined a method to estimate minimum source regions of far fields.
- Obtained results can be approximated quite well via noniterative backpropagation-based pseudospectra, and the multipole theory predictions are consistent with those.
- These representations are sparse, and both computationally tractable and important in practical inverse scattering and imaging.

- Showed such estimates represent probabilistically robust information about the true source support.

- More source support information is encoded in the far fields than one would be inclined to assume, given the nonuniqueness of the inverse problem. It must be incorporated in the inversion algorithms.

Collaborations Welcomed

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My Research Areas

- Wave-based signal processing
- Electromagnetic imaging and inverse problems
- Electromagnetic information theory
- Electromagnetic detection theory
- Electromagnetic theory
- Array signal processing
- Wireless communications

Thank you!