Bayesian Sampling for Error Estimation in Image Reconstruction of X-Ray Radiographs

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Nevada National Security Site
Managed and Operated by National Security Technologies, LLC (NSTec)

The Nevada National Security Site has a long history with different responsibilities over time:

- Atmospheric testing of nuclear weapons (1951–1962)
- Underground nuclear weapons testing (1962–1992)
- Weapons systems experiments (still today)
- Science-base stewardship of nuclear weapons – “Stockpile Stewardship” (still today)

Priscilla, a 37 kiloton balloon test.

The Nevada National Security Site.

Photos courtesy of National Nuclear Security Administration / Nevada Field Office
Abel Inversion – The Problem

Assuming a cylindrically symmetric object, volumetric density is computed by

1. converting the radiograph of intensities to areal density,
2. and applying Abel inversion to convert areal density to volumetric density.
### Abel Inversion – The Deterministic & Stochastic Models

Given radially symmetric density function $x(r)$, the **Abel transform** is given by

$$ b := A(x)(y) = 2 \int_y^R \frac{r \, x(r)}{\sqrt{r^2 - y^2}} \, dr, $$

where $R$ is the maximum radius of the object and $A(x)(y)$ is the areal density of the object.
Abel Inversion – The Deterministic & Stochastic Models

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The discretized, stochastic linear inverse problem is given by

$$b = Ax + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \lambda^{-1}I).$$

areal densities

“true” volumetric density signal

discretized Abel operator

precision parameter
Abel Inversion – Hierarchical Bayesian Sampling

Assumed Distributions

- **Likelihood**: \( b \sim \mathcal{N}(Ax, \lambda^{-1}I) \)
- **Prior**: \( x \sim \mathcal{N}(0, (\delta L)^{-1}) \)
- **Hyperprior**: \( \lambda \sim \Gamma(\alpha, \beta) \)
- **Hyperprior**: \( \delta \sim \Gamma(\theta, \phi) \)
- **Hyperprior**: \( L \sim \text{Wishart}(V, \nu) \)

- Parameters \( \alpha \) and \( \beta \) “replace” the unknown noise precision, \( \lambda \), and tend to be somewhat insensitive.

- Parameters \( \theta \) and \( \phi \) tend to be less sensitive than the parameter they are “replacing”, \( \delta \), which acts as a regularization parameter in the corresponding maximum likelihood estimate.

- We take \( \nu = n \) and \( V \) to be the linearized semi-norm of the total variation regularization solution, which acts to **preserve edges** in this paradigm.
Abel Inversion – The Posterior

Joint Posterior:

\[ p(x, \lambda, \delta, L|b) = \frac{\lambda^{n/2+\alpha-1} \delta^{n/2+\theta-1} \beta \alpha \phi \theta 2^{-\nu/2} |V|^{-\nu/2} |L|^{\nu-n} \Gamma(n(\nu/2))^{-1}}{2\pi^{n} \Gamma(\alpha) \Gamma(\theta)} \times \exp \left( -\beta \lambda - \phi \delta - \frac{1}{2} \left( \lambda ||b - Ax||^2 + \delta x' L x + \text{tr}(V^{-1/2} L) \right) \right) \]

Conditional distributions:

\[ x|\lambda, \delta, L, b \sim \mathcal{N} \left( \lambda(\delta L + \lambda A' A)^{-1} A' b, (\delta L + \lambda A' A)^{-1} \right) \]

\[ \lambda|x, \delta, L, b \sim \Gamma \left( \alpha + n/2, \beta + \frac{1}{2} ||b - Ax||^2 \right) \]

\[ \delta|x, \lambda, L, b \sim \Gamma \left( \theta + n/2, \phi + \frac{1}{2} x'L x \right) \]

\[ L|x, \lambda, \delta, b \sim \text{Wishart} \left( (V^{-1} + \delta xx')^{-1}, \nu + 1 \right) \]

Samples from conditional posteriors are computed using a **Gibbs sampler**. **Sample mean** and **sample standard deviation** characterize the posterior.
Abel Inversion – Results from Synthetic Data
Abel Inversion – Real Data from the NNSS
Abel Inversion – Real Data from the NNSS

TV Reconstruction

MCMC Reconstruction
Abel Inversion – Radiation Transport Simulations

- True Density Profile

Direct Radiation Only

Direct + Scattered Radiation
Abel Inversion – Radiation Transport Simulations

Direct Radiation Only

- TV Reconstruction
- True Density Profile

Direct + Scattered Radiation

- TV Reconstruction
- True Density Profile
Abel Inversion – Radiation Transport Simulations

Direct Radiation Only

Direct + Scattered Radiation
Abel Inversion – Results from Synthetic Data

Characterizing **Stationarity** of the MCMC Chain

Delta Time Series

Lambda Time Series

Signal Times Series (90th element)
Abel Inversion – Radiation Transport Simulations

Characterizing stationarity of the MCMC chain: Direct Radiation Only

[Graphs showing Delta Time Series, Lambda Time Series, and Signal Time Series (90th element)]
Abel Inversion – Radiation Transport Simulations

Characterizing stationarity of the MCMC chain: Direct + Scattered Radiation