DOE/NV/25946--2427

Rigorous Analysis of Noise Correlation

Michał Odyniec, Jerome Blair, National Security Technologies LLC, Livermore, California

Motivation

This paper's aim is threefold:

i. Introduce a rigorous definition for autocorrelation function for signals that do not have instantaneous time value,
ii. Justify the popular expression of autocorrelation used in engineering practice,
iii. Derive relationship between the rigorous the intuitive and the discrete time definitions of signal correlation.

Popular definition:

When a signal has instantaneous values, then its samples and autocorrelation functions are well defined for, respectively discrete and continuous, time as:

Continuous Time :

(1) $$R_y(t,s) := E[y(t)y(s)]$$

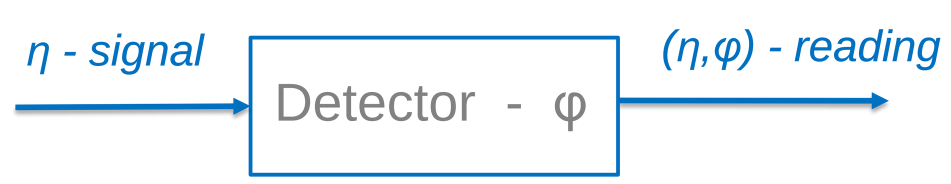
Discrete Time:

(2) $$\sigma_{kl}^2 := E[y_k y_l]$$

where *E(y)* denotes an expected value of the random variable *y*.

Signal value replaced by instrument reading:

When signals do not have instantaneous values, as is typical for noise, we use its measured value (instrument reading):



defined as:

(3) $$(\eta,\varphi) := \int \eta(t)\varphi(t)dt$$

which is a real valued random variable.

The formal integration (3) illustrates , the new random variable as filtered (or smoothened or windowed) noise. Rigorously speaking we model signals of interest as elements of a functional space conjugate to a space of well behaved functions (smooth and fast decreasing test functions) [1-3]. These well behaved functions play the role of instrument responses.

Rigorous definition

Now we use (3) to define the correlation of two random variables:

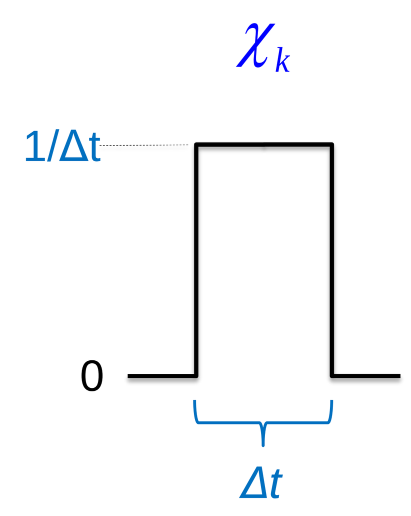
(4) $$R_\eta(\varphi_1,\varphi_2) := E[(\eta,\varphi_1)(\eta,\varphi_2)]$$

that represent two "instrument readings," rather than two moments of time.

Discrete time

Sampled signal– rigorous analysis

Often signal analysis requires translation of continuous time models into discrete time models and vice versa. The time transition is obvious for continuous signals. For signals that do not have instantaneous values, we apply (4) with properly chosen measurement function. We model the "measurement" by a test function χ_k smooth and close to rectangular of value *1/Δt* in the interval *[(k-1/2) Δt, (k+1/2) Δt]* and zero elsewhere.



Then we define the *k-th* signal's sample as:

(5) $$\eta_k := (\eta,\chi_k) \approx \frac{1}{\Delta t}\int_{(k-1/2)\Delta t}^{(k+1/2)\Delta t} \eta(t')dt'$$

In other words, the *k-th* signal's sample is its average taken over the interval *[(k-1/2) Δt, (k+1/2) Δt].*

Correlation Function

Thus, with discrete-time samples defined by (5), the discrete time covariance (2) becomes a special case of (4):

(6) $$\sigma_{kl}^2 = E[\eta_k,\eta_l] = R_\eta(\chi_k,\chi_l)$$

White Noise

A white noise in discrete time is defined by the correlation matrix

(7) $$\sigma_{kk}^2 = \sigma^2 \qquad \sigma_{kl}^2 = 0 \quad \text{for } k \neq l$$

In continuous time its correlation function is often (see [4]) defined as

(8) $$\bar{R}_\eta(t,s) := r^2\delta(t-s)$$

where

(9) $$\sigma^2 = r^2/\Delta t$$

which follows from:

(10) $$\sigma_{kl}^2 = r^2\int \chi_k(t)\chi_l(t)dt = \begin{Bmatrix} r^2/\Delta t & \text{for } k=l \\ 0 & \text{for } k \neq l \end{Bmatrix}$$

In the next section we want to justify the definition (8) and use (4) to prove (9) and (10)

Alternative definition:

Kernel Theorem:

Let us now relate the autocorrelation defined by (4) with an analogue of (1) which is often used in engineering literature [4] and is defined as a two-variable distribution :

(11) $$\bar{R}_\eta(t,s) := E[\eta(t)\eta(s)]$$

Intuitively, i.e., without concern for the existence of the integrals, one formally derives :

$$E[(\eta,\varphi_1)(\eta,\varphi_2)] = E\left[\iint \eta(t)\eta(s)\varphi_1(t)\varphi_2(s)dtds\right] =$$
$$= \iint E[\eta(t)\eta(s)]\varphi_1(t)\varphi_2(s)dtds$$

That suggests the definition (11):

(12) $$R_\eta(\varphi_1,\varphi_2) = \iint \bar{R}_\eta(t,s)\varphi_1(t)\varphi_2(s)dtds$$

Rigorously, the existence of $\bar{R}_\eta$ follows from the Kernel Theorem [1-3]

When the signal $\eta(t)$ is continuous , then (11) is well defined and when the test functions φ are narrowly focused (the measurements are focused in time) small, then (4) approaches (11)=(1).

It follows from (12) that the relation of the kernel distribution $\bar{R}_\eta$ to the discrete time covariance is :

(13) $$\sigma_{kl}^2 = R_\eta(\chi_k,\chi_l) = (\Delta t)^{-2}\iint \bar{R}_\eta(t,s)dtds$$

where the integration goes over intervals of length *Δt.*

We can now derive (10) as a special case of (13):

$$\sigma_{kl}^2 = R_\eta(\chi_k,\chi_l) = \iint \bar{R}_\eta(t,s)\chi_k(t)\chi_l(s)dtds$$
$$= r^2\iint \delta(t-s)\chi_k(t)\chi_l(s)dtds = r^2\int \chi_k(t)\chi_l(t)dt$$

References:

[1] L. Schwartz, *Theorie des distributions*, Paris: Hermann, 1966.
[2] A. M. Yaglom, *Introduction to the Theory of Stationary Random Functions*, Mineola, NY : Dover, 2004.
[3] M. Gelfand, N. Ya. Vilenkin, *Generalized Functions*, vol. 4. Waltham, MA: Academic Press, 1964.
[4] A. Papoulis, *Probability, Random Variables, Stochastic Processes*, McGraw-Hill, Inc., 1991.

This poster was prepared for and presented at the 19th CASIS workshop on Signal and Image Sciences on May 13, 2015, by Michał Odyniec. This work was done by National Security Technologies, LLC, under Contract No. DE-AC52-06NA25946 with the U.S. Department of Energy. DOE/NV/25946--2427