Two Layer Foldy-Lax Scattering Model

Sean K. Lehman, Ph.D.

Lawrence Livermore National Laboratory

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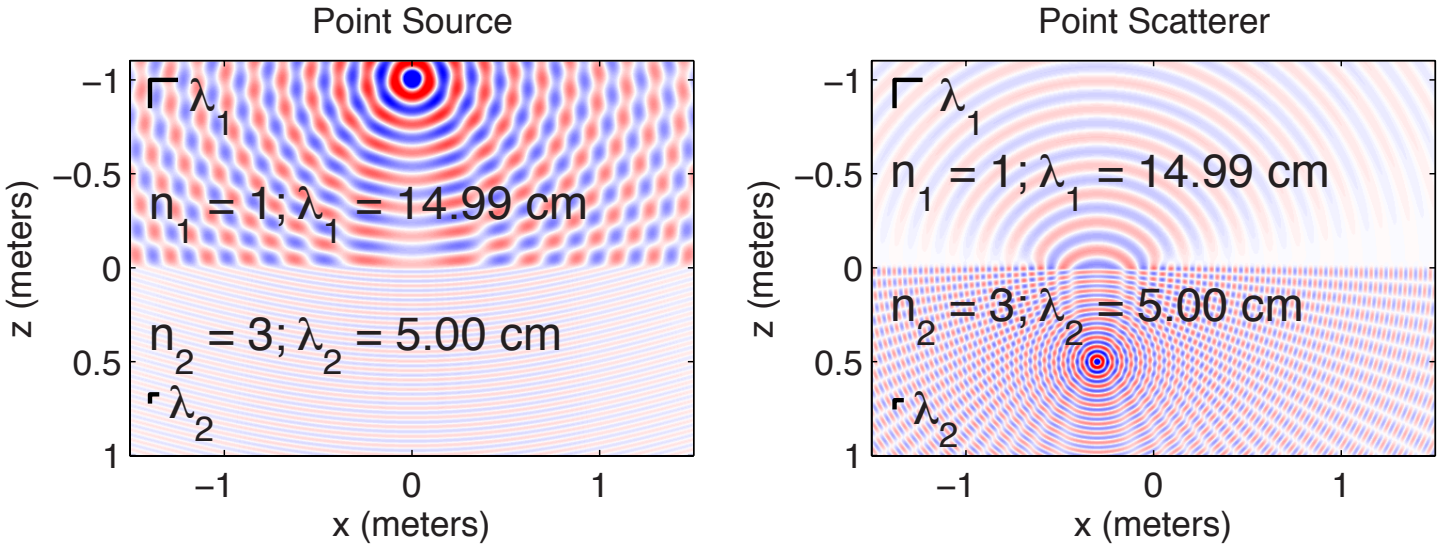

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"Scary incantations in the wrong hands..."
— From the Collide song *Falling Up*

Foldy-Lax Theory

Foldy-Lax theory provides a model of (frequency domain) wave scattering for a collection of point scatterers

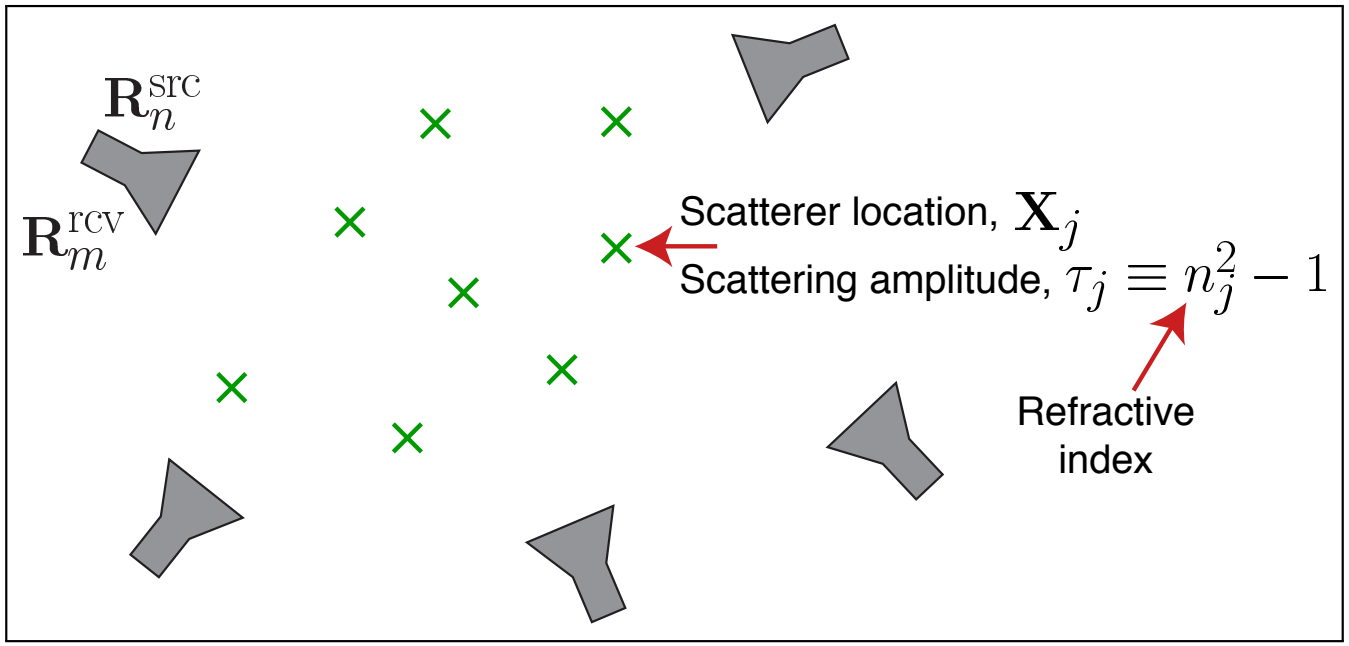
- M. Lax, "Multiple Scattering of Waves", *Reviews of Modern Physics*, October 1951, **23**(4), pp287–310
- L. L. Foldy, "The Multiple Scattering of Waves: I. General Theory of Isotropic Scattering by Randomly Distributed Scatterers", *Pysical Review*, February 1945, **67**(3 & 4), pp107–119
- M. Lax, "The Multiple Scattering of Waves: II. The Effective Field in Dense Systems", *Physical Review*, February 1952, **85**(4), pp621–629

Foldy-Lax Theory in a Nutshell – I

The integral form of the Helmholtz equation

$$\psi^{\text{scat}}(\mathbf{r}, \mathbf{R}_n^{\text{src}}, \omega) = k_0^2(\omega) \int d\mathbf{r}' \; G(\mathbf{r}, \mathbf{r}', \omega) \; o(\mathbf{r}') \; \psi^{\text{tot}}(\mathbf{r}', \mathbf{R}_n^{\text{src}}, \omega),$$

specialized to point scatterers is



$$\boxed{\psi^{\text{scat}}(\mathbf{r}, \mathbf{R}_n^{\text{src}}, \omega) = k_0^2(\omega) \sum_j \tau_j \; G(\mathbf{r}, \mathbf{X}_j, \omega) \; \psi^{\text{tot}}(\mathbf{X}_j, \mathbf{R}_n^{\text{src}}, \omega)}$$

Foldy-Lax Theory in a Nutshell – II

The total field as observed at the scatterers must be known:

$$\psi^{\mathrm{scat}}(\mathbf{r}, \mathbf{R}_n^{\mathrm{src}}, \omega) = k_0^2(\omega) \sum_j \tau_j \; G(\mathbf{r}, \mathbf{X}_j, \omega) \; \boxed{\psi^{\mathrm{tot}}(\mathbf{X}_j, \mathbf{R}_n^{\mathrm{src}}, \omega)}$$

The total field due to source n is given by

$$\begin{aligned} \psi^{\mathrm{tot}}(\mathbf{r}, \mathbf{R}_n^{\mathrm{src}}, \omega) &= \psi^{\mathrm{inc}}(\mathbf{r}, \mathbf{R}_n^{\mathrm{src}}, \omega) + \\ & \quad k_0^2(\omega) \sum_j \tau_j \; G(\mathbf{r}, \mathbf{X}_j, \omega) \; \boxed{\psi^{\mathrm{tot}}(\mathbf{X}_j, \mathbf{R}_n^{\mathrm{src}}, \omega)} \end{aligned}$$

Evaluate this at $\mathbf{r} = \mathbf{X}_j$ (excluding the self-scattering terms where the Green function diverges):

$$\begin{aligned} \boxed{\psi^{\mathrm{tot}}(\mathbf{X}_j, \mathbf{R}_n^{\mathrm{src}}, \omega)} &= \psi^{\mathrm{inc}}(\mathbf{X}_j, \mathbf{R}_n^{\mathrm{src}}, \omega) + \\ & \quad k_0^2(\omega) \sum_{j' \neq j} \tau_{j'} \; G(\mathbf{X}_j, \mathbf{X}_{j'}, \omega) \; \psi^{\mathrm{tot}}(\mathbf{X}_{j'}, \mathbf{R}_n^{\mathrm{src}}, \omega) \end{aligned}$$

Foldy-Lax Theory in a Nutshell – III

We cast this into the form of a matrix equation,

$$\psi^{\mathrm{tot}}(\mathbf{X}_j, \mathbf{R}_n^{\mathrm{src}}, \omega) -$$

$$k_0^2(\omega) \sum_{j' \neq j} \tau_{j'} \; G(\mathbf{X}_j, \mathbf{X}_{j'}, \omega) \; \psi^{\mathrm{tot}}(\mathbf{X}_{j'}, \mathbf{R}_n^{\mathrm{src}}, \omega) \quad = \quad \psi^{\mathrm{inc}}(\mathbf{X}_j, \mathbf{R}_n^{\mathrm{src}}, \omega),$$

$$\underbrace{\left[I - k_0^2(\omega)\mathrm{G}(\omega)\mathrm{T}\right]}_{A} \underbrace{\bar{\psi}_n^{\mathrm{tot}}(\omega)}_{x} \quad = \quad \underbrace{\bar{\psi}_n^{\mathrm{inc}}(\omega)}_{b}$$

solve

$$\bar{\psi}_n^{\mathrm{tot}}(\omega) \quad = \quad \left[I - k_0^2(\omega)\mathrm{G}(\omega)\mathrm{T}\right]^{-1} \; \bar{\psi}_n^{\mathrm{inc}}(\omega)$$

and back substitute,

$$\psi^{\mathrm{scat}}(\mathbf{r}, \mathbf{R}_n^{\mathrm{src}}, \omega) = k_0^2(\omega) \sum_{j} \tau_j \; G(\mathbf{r}, \mathbf{X}_j, \omega) \; \psi^{\mathrm{tot}}(\mathbf{X}_j, \mathbf{R}_n^{\mathrm{src}}, \omega)$$

Aside on Stability

For a non-diverging solution, require

$$k_0^2 \left\| G(\omega) T \right\|_2 < 1$$

This bounds the scattering amplitudes,

$$|\tau| < \frac{4\pi}{(J-1)\ k_0^2(\omega)}$$

where $R_{\min}$ is the minimum scatterer separation and J is the number of scatterers.

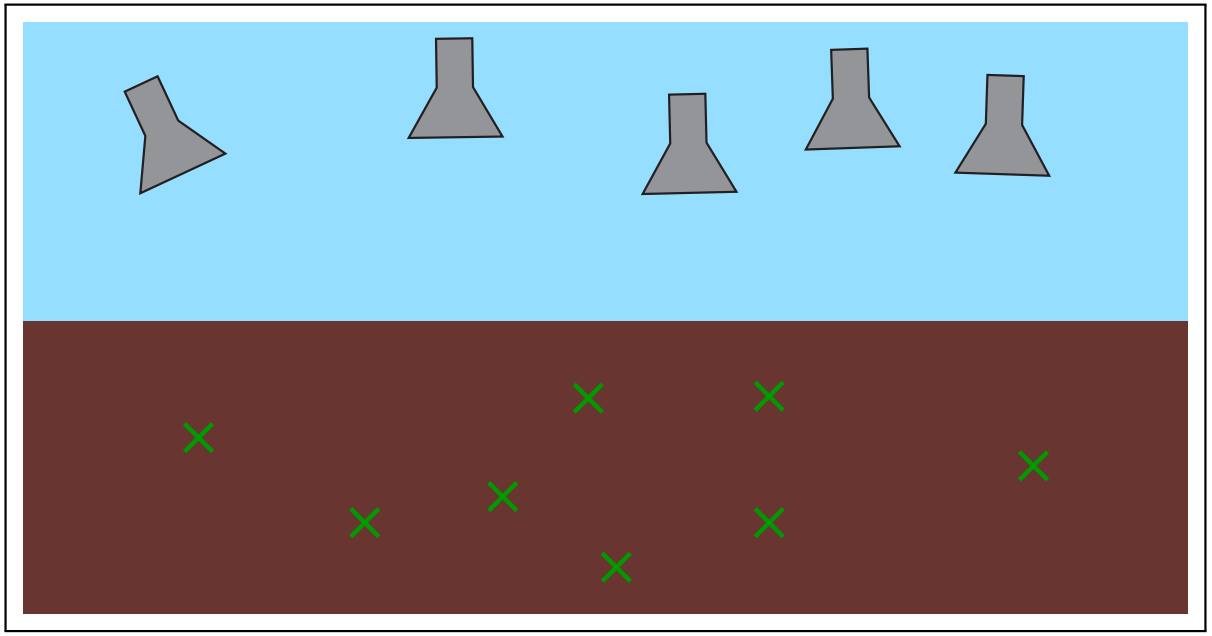
The Green Function

Nothing has been said of the Green function.

$$\bar{\psi}_n^{\mathrm{tot}}(\omega) = \left[I - k_0^2(\omega)\, \mathrm{G}(\omega)\, \mathrm{T} \right]^{-1} \bar{\psi}_n^{\mathrm{inc}}(\omega)$$

$$\psi^{\mathrm{scat}}(\mathbf{r}, \mathbf{R}_n^{\mathrm{src}}, \omega) = k_0^2(\omega) \sum_j \tau_j \; G(\mathbf{r}, \mathbf{X}_j, \omega) \; \psi^{\mathrm{tot}}(\mathbf{X}_j, \mathbf{R}_n^{\mathrm{src}}, \omega)$$

Consider a two planar layer environment:



Sommerfeld Integral

Deriving the Green function, $G(\mathbf{r}, \mathbf{X}_j, \omega)$, for a two planar layer medium requires solving a Sommerfeld integral such as,

$$G(\mathbf{r}_\perp, z) = \frac{i}{2(2\pi)^2} \int d\mathbf{k}_\perp \, \frac{1}{\gamma_0(\mathbf{k}_\perp, \omega)} \, e^{i\gamma_0(\mathbf{k}_\perp, \omega)z} \, e^{i\mathbf{k}_\perp \cdot \mathbf{r}_\perp}$$

where

$$\gamma_0(\mathbf{k}_\perp, \omega) = \sqrt{k_0^2(\omega) - \mathbf{k}_\perp^2}$$

which is difficult.

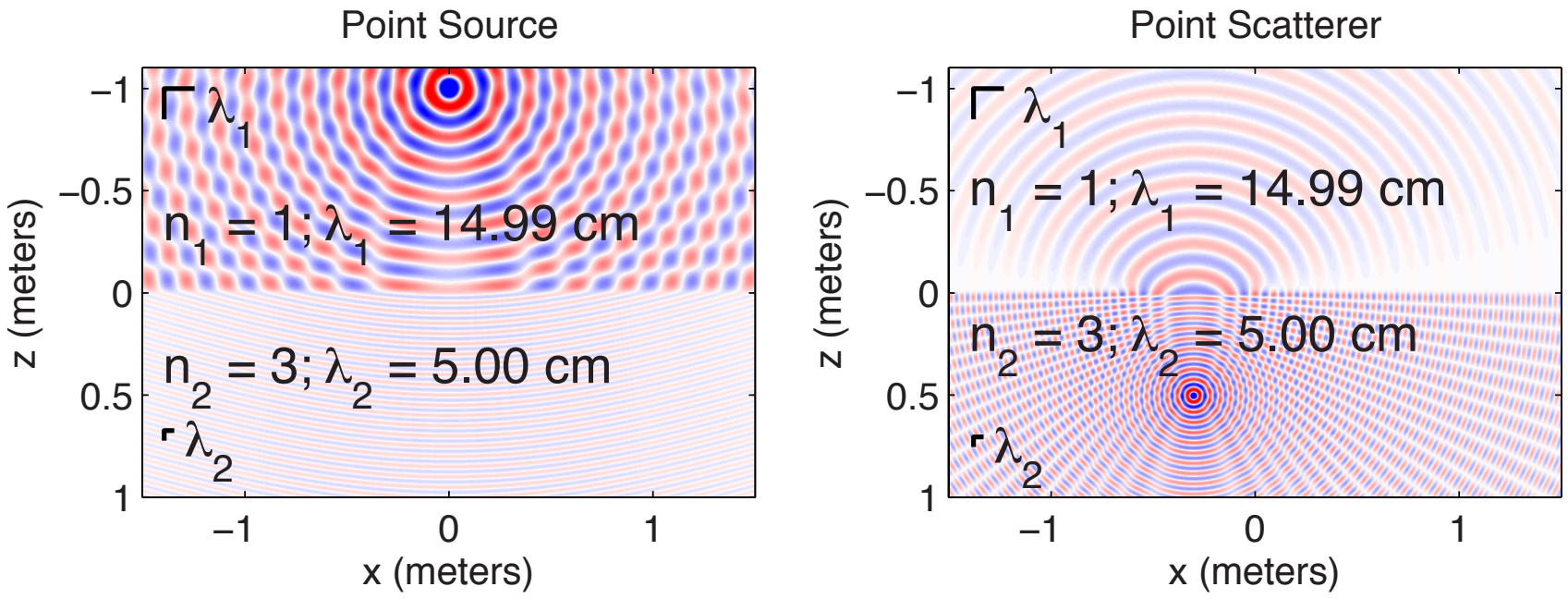
A Fast Sommerfeld Integral Solver

- A. Hochman, "A Numerical Methodology for Efficient Evaluation of 2D Sommerfeld Integrals in the Dielectric Half-Space", *IEEE Trans. Antennas and Propagation*, February 2010, **58**(2), pp413–431.
- MATLAB function, "Numerically-Determined Steepest-Descent Path (NDSDP)": `ndsdp()`
- `http://web.mit.edu/hochman/www/`
- Put the `ndsdp()` function into the planar layer Green function
- Put the Green function into the Foldy-Lax solver

The result is a fast-ish MATLAB two-layer Foldy-Lax solver: It must perform two `ndsdp()` calls for each scatterer for each frequency.

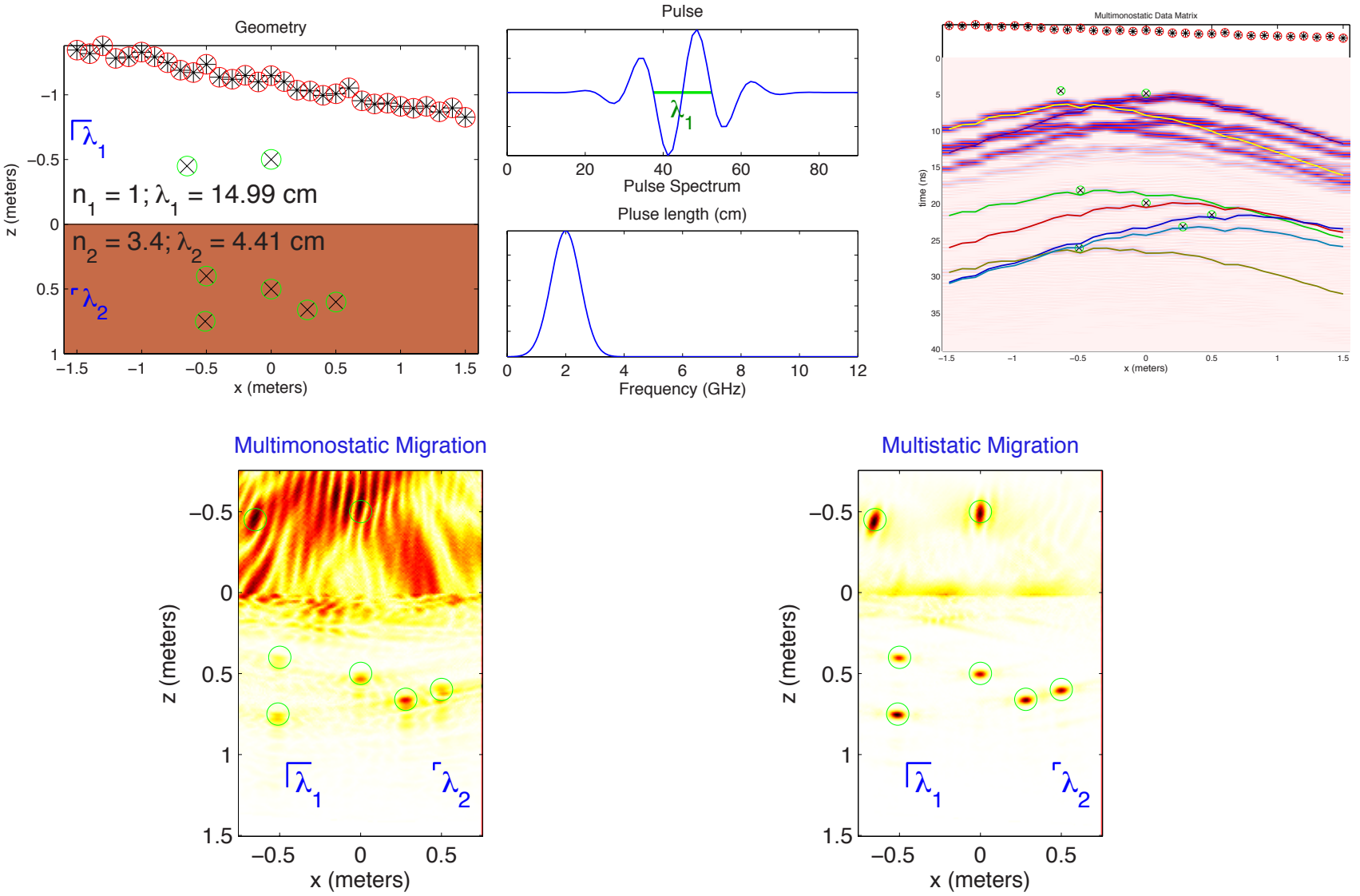
Numerically-Determined Steepest-Descent Path

For each frequency, Numerically-Determined Steepest-Descent Path (`ndsdp.m`) solves,

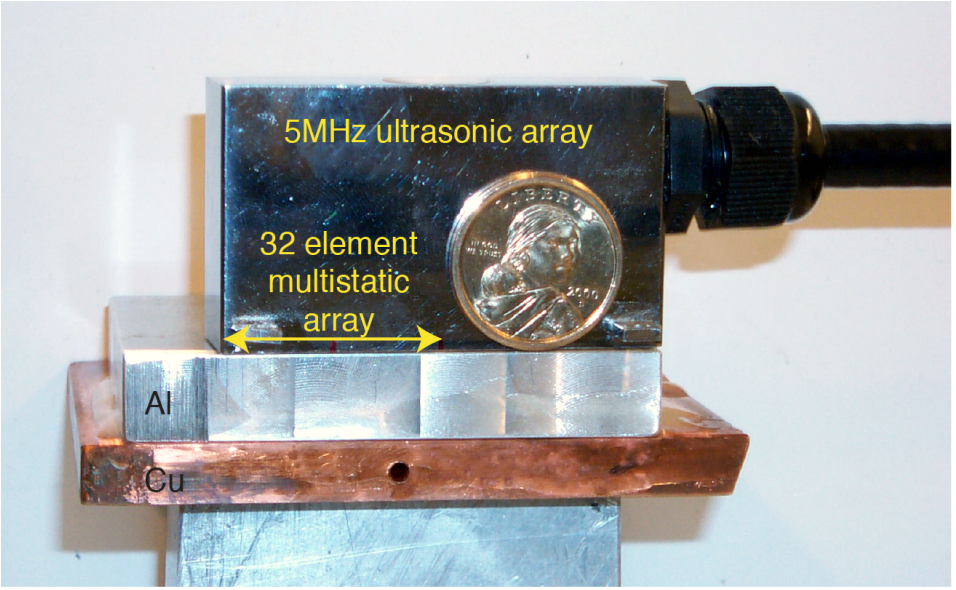


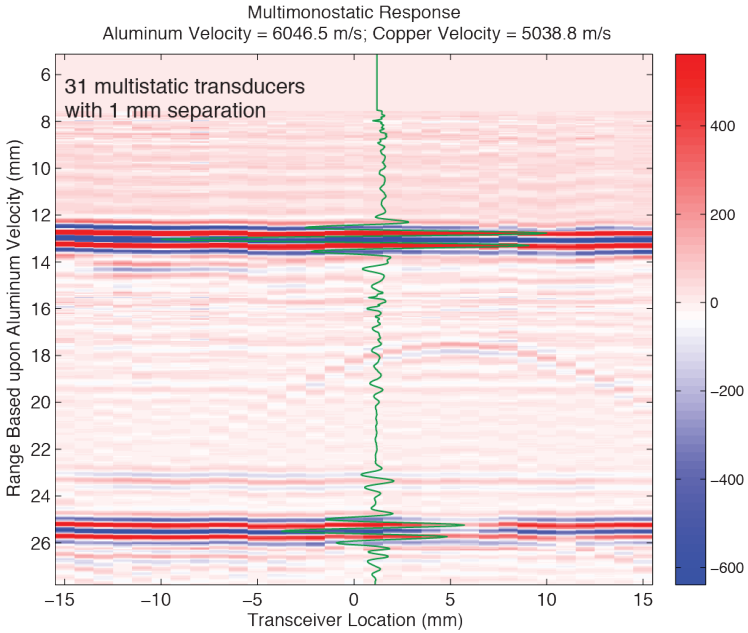
Examples

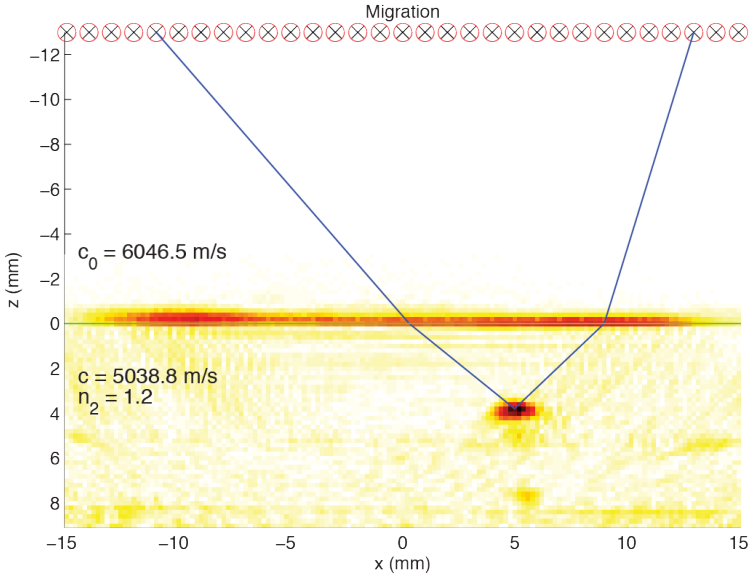
Simulated Verification



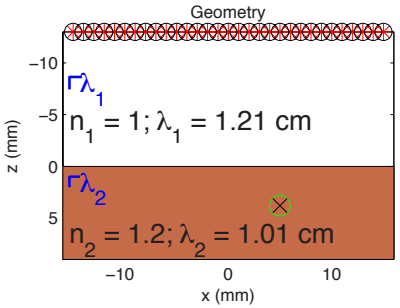
Aluminum/Copper Layer

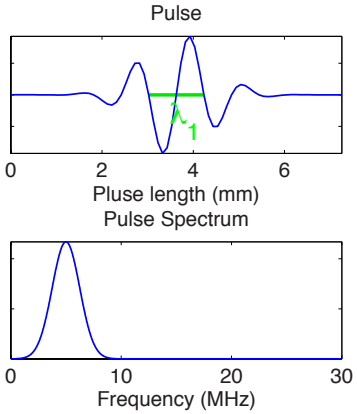


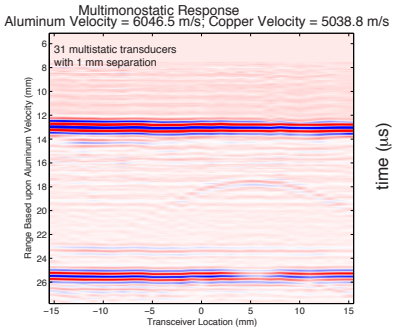


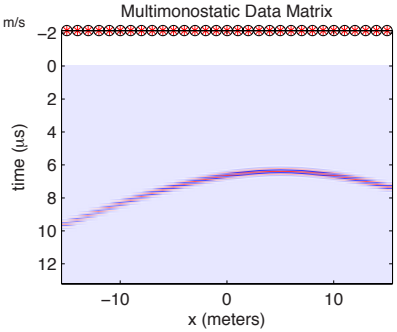


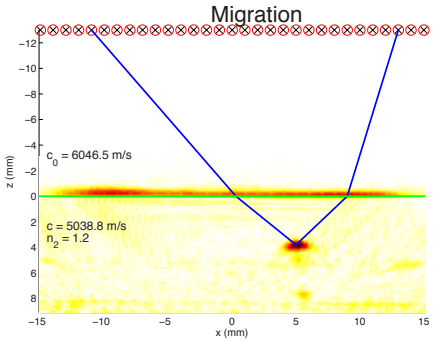
Aluminum/Copper Layer

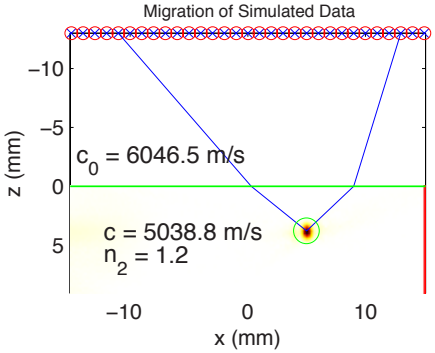




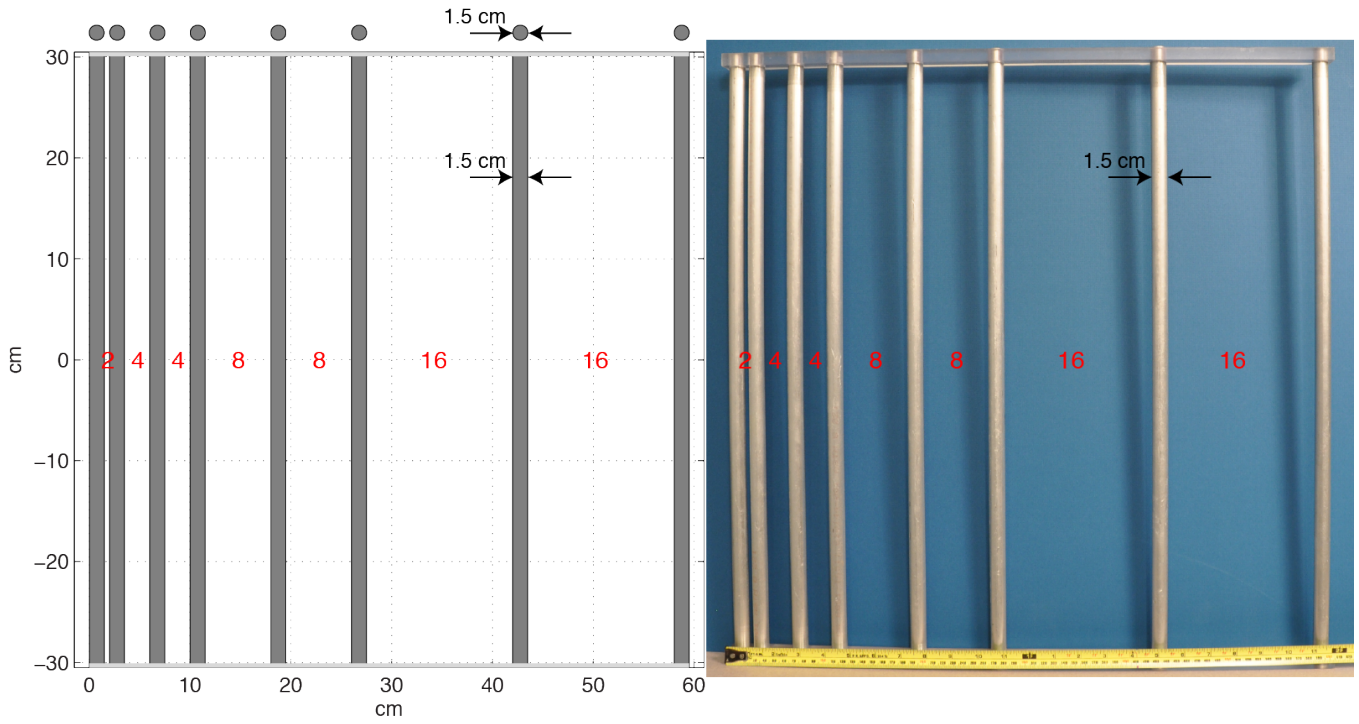


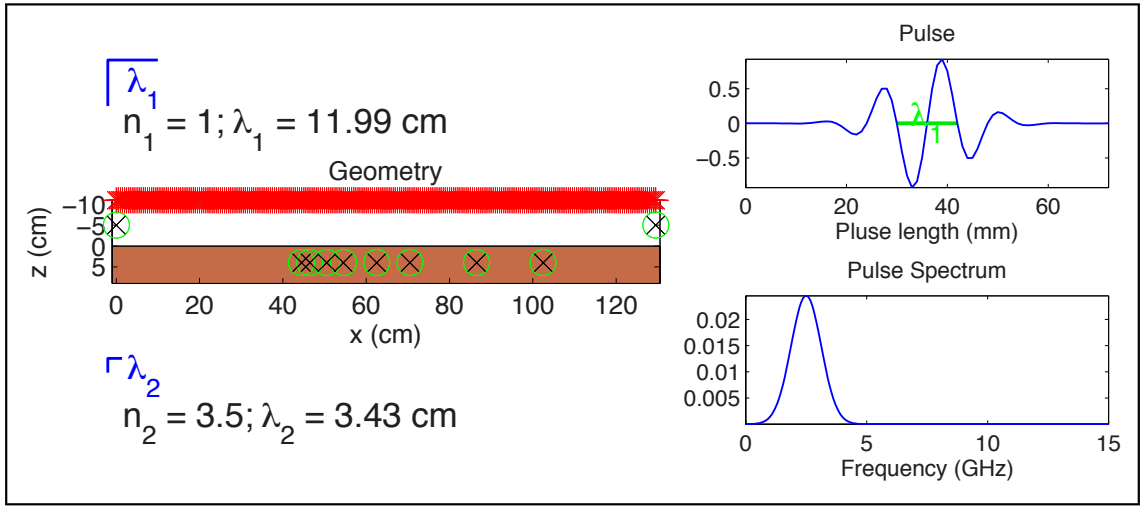






Buried Resolution Phantom





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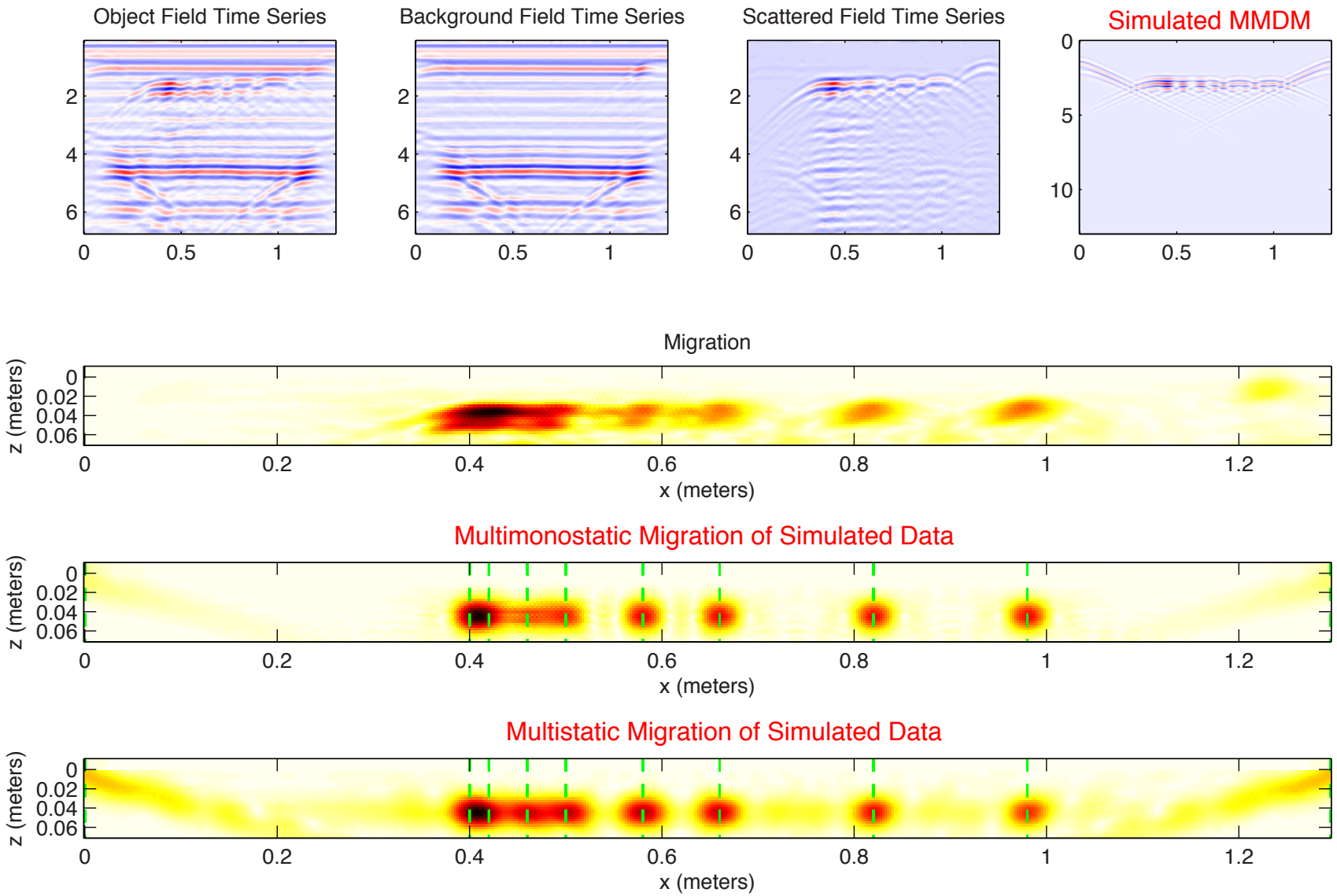

Summary

- Foldy-Lax is a full point scatterer scattering theory with the only concession being that of no self-scattering.
- The theory is derived independently of the Green function.
- A Sommerfeld integral solver is used to create a two-layer Green function for a Foldy-Lax scattering code.