Rigorous Analysis of Poisson Process Detection

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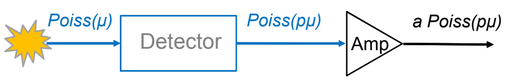
Motivation

This paper is motivated by an NToF analysis in which the neutrons created in D + T reaction pass through detector(s), filter(s) and amplifier(s). They are modeled by a counting Poisson process, denoted by $x=Poiss(\mu)$, with the Poisson constant μ which equals the process's expected value and also its variance $\mu=\langle x\rangle=V(x)$.



It turns out that a stochastic signal (in our case the particle counting (Poisson) process) ends up with different statistical properties after passing through a detector as opposed to passing through an amplifier (or attenuator). Our aim is to provide a rigorous model for the detection mechanism of Poisson processes as opposed to amplification/attenuation mechanisms.

Deterministic system - amplifier/attenuator

For an amplifier described by $y=a\,x$

we have $\langle y\rangle=a\langle x\rangle$ and $V(y) = a^2 V(x)$.

Stochastic system - detector

Detectors produce an output to an incoming particle with a certain probability 'p' (detection efficiency). At first sight the mechanism seems to be similar to that of $y=ax$. We show here that such detector differs from an attenuator *with* $a=p$.

Poisson process

Consider a signal modeled by a Poisson process with the Poisson constant such a process may describe:

i. radioactive decay,
ii. analysis of queueing phenomena (arrival on phone calls to an exchange or signals in computer communication, arrival of department store customers at a cashier, departing planes on a runway),
iii. analysis of any sort of accidents (machine failures, breakdowns, errors, etc.),
iv. shot noise in electronic devices.

Let $x=Poiss(\mu)$ be a Poisson process, the probability that $x = k$ equals:

$$p_{xk} := P\left(\{x=k\}\right) = \frac{\mu^k}{k!}e^{-\mu}$$

Its generating function equals:

$$X(s) := \langle s^x\rangle = \sum_{k=0}^{\infty} s^k P\left(\{x=k\}\right) = \sum_{i=0}^{\infty} p_{xk}s^k$$

Substituting p_{xk} we get the explicit formula for Poisson generating function:

$$X(s) = \sum_{k=0}^{\infty}\frac{\mu^k}{k!}e^{-\mu}s^k = e^{\mu s}e^{-\mu} = e^{\mu(s-1)}$$

Main Result

Particle detector response – rigorous analysis

Consider (nonnegative) integer-valued random variables *x*, *g*, *y*, with probability distributions: $\{p_{x0},p_{x1},\ldots\}$, $\{p_{y0},p_{y1},\ldots\}$, $\{p_{g0},p_{g1},\ldots\}$. Let *y* be a sum of '*x*' identically distributed random variables *g* . We want to find *y* in terms of *x* and *g*.

$$y = \sum_{i=0}^{x} g_i \qquad \textbf{(0)}$$

We can think about *x* as the number of particles arriving at a (generalized) "detector"; about *g* as the detector's response to a single particle, and about y as the detectors response (the number of particles leaving the detector).

A simple detector with the detection probability *p* is a special case of the above with $\{p_{g0},p_{g1},\ldots\} = \{1-p,p\}$.

The corresponding generating functions are:

$$X(s) := \sum_{i=0}^{\infty} p_{xi}s^i \quad G(s) := \sum_{i=0}^{\infty} p_{gi}s^i \quad Y(s) := \sum_{i=0}^{\infty} p_{yi}s^i$$

When *x* is the Poisson process $x=Poiss(\mu)$, its generating function equals:

$$X(s) = e^{\mu(s-1)}$$

When *g* is the Bernoulli process $\{p_{g0},p_{g1},\ldots\} = \{1-p,p\}$, its generating function is:

$$G(s) = g_0 + g_1 s = (1-p) + ps$$

Now, we apply the property (proven in the right column)

$$Y(s) := X\left(G(s)\right) \qquad (1)$$

to Poisson and Bernoulli processes:

$$Y(s) = X\left(G(s)\right) = e^{\mu(G(s)-1)} = e^{\mu((1-p)+ps-1)} = e^{\mu p(s-1)}$$

Consequently

$$Y(s) = e^{\mu p(s-1)} \qquad (2)$$

therefore *y* is a Poisson process $y = Poiss(\mu p)$
with $\langle y\rangle = p\langle x\rangle = \mu p = p\,V(x) = V(y)$.

This result is known in theory of random processes [1,2]. However the published proofs are either sketchy [1] or abstract [2]. Here we present a detailed and rigorous proof in the language of elementary probability in hope of making it easily accessible.

[1] W. Feller "An Introduction to Probability Theory...", J. Wiley, 1961.

[2] J.F.C. Kingman "Poisson Processes", Oxford, 2002

Detailed Proof of (1): $Y(s) := X\left(G(s)\right)$

The trick is to replace the random sum of (0) by definite sums.

From probability definition:

$$p_{yk} := P\left(\{y=k\}\right) = \sum_{n=0}^{\infty} P\left(\{y=k\}\cap\{x=n\}\right)$$

$$= \sum_{n=0}^{\infty} P\left(\{y=k\}\,|\,\{x=n\}\right)P\left(\{x=n\}\right)$$

where (again from probability definition):

$$P\left(\{x=n\}\right) = p_{xn}$$

$$P\left(\{y=k\}\,|\,\{x=n\}\right) = P\left(\left\{\sum_{i=0}^{n} g_i = k\right\}\right)$$

Consequently

$$p_{yk} = \sum_{n=0}^{\infty} P\left(\left\{\sum_{i=1}^{n} g_i = k\right\}\right)p_{xi}$$

Therefore, the generating function of *y* equals:

$$Y(s) = \sum_{k=0}^{\infty} p_{yk}s^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} P\left(\left\{\sum_{i=1}^{n} g_i = k\right\}\right)p_{xn}s^k =$$

$$= \sum_{n=0}^{\infty} p_{xn}\underbrace{\sum_{k=0}^{\infty} P\left(\left\{\sum_{i=1}^{n} g_i = k\right\}\right)s^k}_{\text{Generating function of the sum } S_n := \sum_{i=1}^{n} g_i} = \sum_{n=0}^{\infty} p_{xn}G(s)^n$$

Indeed, recall that the generating function is defined by the expected value:

$$G(s) := \langle s^g\rangle = \sum_{i=0}^{\infty} s^i P\left(\{g=i\}\right) = \sum_{i=0}^{\infty} s^i p_{gi}$$

thus, the generating function of a sum of *n* iid random variables $S_n := \sum_{i=1}^{n} g_i$ equals:

$$\langle s^{g_1+\ldots+g_n}\rangle = \langle s^{g_1}\rangle\cdot\ldots\cdot\langle s^{g_n}\rangle = \langle s^{g_k}\rangle^n = G(s)^n$$

That yields

$$Y(s) = \sum_{n=0}^{\infty} p_{xn}G(s)^n = X\left(G(s)\right)$$

which proves (1).

This poster was prepared for and presented at the 18th CASIS workshop on Signal and Image Sciences on May 21, 2014, by Michał Odyniec. This work was done by National Security Technologies, LLC, under Contract No. DE-AC52-06NA25946 with the U.S. Department of Energy and supported by the Site-Directed Research and Development Program.
DOE/NV/25946--2090